The general type of the reduced equation, though certainly not its very simple form, could have been foretold from the results at the beginning of section 2.

If we consider the transformed equation in the form (14), and use (12), where y_i is now a root of (14), we can express the roots of the original cyclic cubic (13) as follows:

$$x_i = [p + (-3)^{1/2}]\frac{y_i + 2(-3)^{1/2}p + 6}{y_i + 2(-3)^{1/2}p - 6}.$$

5. **Rational Cubics with a Rational Root.** It is easily seen that if a cubic of form (1) has a rational root, it must be of the type $x^3+(n-r^2)x-rn=0$, where r is the rational root and n is also rational. In this case we have

THEOREM IV. *Any rational cubic with a rational root can be transformed, by a rational Tschirnhaus transformation, into an equation of the form* $y^3+sy=0$, *where s is rational.*

The procedure in this case is somewhat different, for we are now seeking to make $\sum y^3$ equal to zero. If we substitute $a_2=n-r^2$, $a_3=-rn$, in (7), the equation $\sum y^3=0$ becomes

$$3rnk_2^3+2(n-r^2)^2k_2^2-3rn(n-r^2)k_2+3r^2n^2+(2/9)\ (n-r^2)^3=0.$$

It can be easily verified, though less easily derived, that this equation has the rational root $k_2=(r^2+2n)/(-3r)$. We note that if r is zero the given equation already has the required normal form, and no transformation is necessary. If we substitute this value of k_2 in (6) and take $m=1$, $\sum y^2$ reduces to $(2/9)r^{-2}(2r^2+n)^2(r^2-4n)$, and the transformed equation is $y^3-(1/9)r^{-2}(2r^2+n)^2\ (r^2-4n)y=0$, which is of the desired form.

BENJAMIN PEIRCE'S LINEAR ASSOCIATIVE ALGEBRA AND C. S. PEIRCE

By RAYMOND CLARE ARCHIBALD, Brown University

Since the greater part of an issue of this Monthly (January, 1925) was devoted to the life and work of Benjamin Peirce, it would seem appropriate to place on record in the same publication a vigorous document of his very able son, the late C. S. Peirce. This document, dated June 28, 1910, is a two page sheet which was in his copy of Jordan's *Traité des Substitutions et des Equations Algébriques*. It was written by Peirce in his seventy-first year. Except for the footnotes, which I have added, the transcription of the document is as follows:

"I will record a reminiscence about this book. It was published in 1870, the same year as the date of the original edition of my father's *Linear Associative*

Algebra (though I am sure this was not lithographed for a year or two after the general theory was complete). I had first put my father up to that investigation by persistent hammering upon the desirability of it. There was one feature of this work, however, which I never could approve of, and in vain endeavoured to get him to change. It was his making his coefficients, or scalars, to be susceptible of taking imaginary values. In vain I represented to him that the system of imaginary quantity has two dimensions, and is consequently a double algebra. But it was always next to impossible to induce him to take a logical view of any subject. He did me the honor to reply to my arguments in a footnote on p. 19 of the Ed. of 1870 (p. 9 of that of 1882).[1] The reply is pure bosh. His "broad philosophy" which could not be definitely expressed, was a mere habit of feeling. He was a creature of feeling, and had a superstitious reverence for "the square root of minus one"; and as to the absence of it "trammeling" research, that only means that he was not in possession of any machinery for dealing with the problems that lie beyond its scope. If Hamilton had done as he would have had him, the calculus of quaternions could not have come into being, because division would not generally have had a determinate result. The substance of this work of Jordan was inaccessible to mathematicians who did not choose to devote near a life-time to it, before the work appeared, but if my father had been able to acquaint himself with the Galois theory of equations, and had taken advantage of the possibility, he certainly must have come to see that I had been quite right in my contention. I happened to be in London in 1870 and coming across the book on Hachette's counter purchased it. But I turned it over to the Coast Survey and so was subsequently forced to surrender it along with much else that maimed me intellectually. The dirty fellows who played me this trick got nothing by it except the pleasure of harming me. My activities not lying in the direction of mathematics, I never, while I had the book, got time to master it. When I found that my brother[2] had purchased the book in 1874, I told him it was the very book he needed to study, and that he would get a flood of illumination

[1] C. S. Peirce was the editor, with "addenda and notes," of the second edition of his father's *Linear Associative Algebra*. The footnote here referred to is as follows: "Hamilton's total exclusion of the imaginary of ordinary algebra from the calculus as well as from the interpretation of quaternions will not probably be accepted in the future development of this algebra. It evinces the resources of his genius that he was able to accomplish his investigations under these trammels. But like the restrictions of the ancient geometry, they are inconsistent with the generalizations and broad philosophy of modern science. With the restoration of the ordinary imaginary, quaternions becomes Hamilton's biquaternions. From this point of view, all the algebras of this research would be called bi-algebras. But with the ordinary imaginary is involved a vast power of research, and the distinction of names should correspond; and the algebra which loses it should have its restricted nature indicated by such a name as that of *semi-algebra*."

[2] J. M. Peirce, for forty-five years professor of mathematics at Harvard University.

from it. But he only cut the leaves of the first sheet, and remained to his dying day a superstitious worshipper of two hostile gods, Hamilton and the scalar $\sqrt{(-1)}$. As a professor of mathematics, one would have thought he might have fancied getting some insight to the mathematical advances of his day, most of which have involved the influence of this work; but that wasn't his nature. He was also largely a creature of feeling though his feelings were not of the violent kind. When he died, he left me, as his sole but sufficient legacy (I being the only poor member of the family) his mathematical books, having previously dispossessed himself of every one that he knew I particularly desired. He thought I had a copy of this."

July, 1927.

QUESTIONS AND DISCUSSIONS

EDITED BY H. E. BUCHANAN, Tulane University, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS

I. A GENERALIZATION OF THE STROPHOID

By F. H. HODGE, Purdue University

We start with a circle of radius $a+k$ tangent to the y-axis at the origin and a fixed point A with coordinates $(a, 0)$. A line through A meets the circle in the points P and P'. We lay off on this line $MP=PN=OP$ and $M'P'=P'N'=OP'$. (See Fig. 1) The locus traced by M, N, M', and N' as the line rotates about A (See Fig. 2) is the curve that we are to consider.

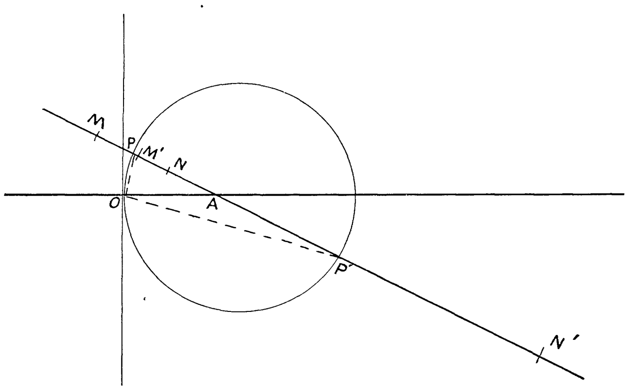

FIG. 1.