attempt to go through the traditional 'problems of ontology' one-by-one, examine the premises which generate the problem, and see whether there is any reason to believe these premises." Thus Rorty proposes an historic twist to the linguistic turn in recent philosophy.

In sum, despite the merits of individual essays, there is need for a sequel to *Contemporary American Philosophy, First Series* (1930). Smith's volume does not meet this need.

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[1]See my review of Black's book, "Philosophies in America," *Southern Journal of Philosophy* IV (1966), 79-80.

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Charles S. Peirce as Mathematical Geodesist

I

The figure of the earth has been a concern of mathematical geodesy ever since Eratosthenes of Cyrene in ancient times estimated its circumference by an astronomical-geodetical method. An assumption that the earth was spherical in form was called in question in the 17th century as measurements of arcs of meridians gave variable results. The astronomer Jean Richer made a decisive discovery in 1672 when he observed that his pendulum-clock beat more slowly in equatorial Cayenne than in Paris. This phenomenon was explained by Newton's deduction from the theory of gravitation that the earth is an oblate spheroid which is flattened at the poles. Since the frequency of oscillation of a pendulum depends upon the intensity of gravity, this demonstration of variation with position provided new means for determining the figure of the earth from surveys of gravity on the surface. The earth was conceived as an ellipsoid of revolution and its departure from sphericity was expressed in terms of the ellipticity, or flattening, or compression, of the earth. If a is the polar semiaxis, and b the equatorial semiaxis, the ellipticity may be defined as (a-b)/a. Its determination is a principal problem of geodesy.

The United States Coast and Geodetic Survey has made important contributions to geodesy by surveys of gravity and the developments of theory. Charles S. Peirce, as a member of the Survey in the 19th century, contributed notably to the practice and theory of research in gravity. Peirce is best known in the field for his measurements of the intensity of gravity by pendulums at stations in America and Europe. But he also contributed to the method of utilizing the results of surveys of gravity for the determination of the figure of the earth. In "Introduction historique," *Mémoires sur le Pendule*, by C. Wolf, Paris, 1889, there is given a table of values of ellipticity, as determined by mathematicians from Newton (1713) to Helmert (1884). The table includes a value calculated by Charles S. Peirce in 1881 and published in his memoir, "On the deduction of the ellipticity of the earth from pendulum experiments."[1]

II

A calculation of the ellipticity of the earth is based upon data which are obtained by physical measurements of absolute or relative values of the intensity of gravity at stations specified in terms of latitude, longitude, elevation above sea level, and topographical characteristics. Intensity of gravity may be expressed in various ways: as the length of the pendulum that beats seconds, the seconds pendulum; as the acceleration of a freely falling body; as the number of oscillations per diem of a pendulum that beats seconds at a reference station. Since relative gravity could be determined more accurately than absolute gravity, Peirce held that the modes of representation should be different. In view of a preference for the expression of gravity in terms of number of oscillations per diem, he made a convenient modification of it. Peirce observed that a change of one beat per day in the frequency of a pendulum that beats seconds at the equator corresponds to a change of one unit in the fifth place of the seven-place common logarithm of the length of the pendulum. In view of this circumstance, he expressed relative gravity in terms of station number, N, which was defined by

$$N = 100{,}000 \log g + A,$$

where A is such a constant that the average value of N for all stations is 86,400, the number of beats of the seconds pendulum per mean solar day. The unit of N was called the logarithmic second (log. sec.). When the station number was corrected for elevation and latitude, the excess (positive or negative) of the reduced station number over 86,400 was called the station-error. Peirce found the expression of gravity in terms of logarithmic seconds a great convenience in computation, since multiplications and divisions could be carried out by additions and subtractions without reference to tables of logarithms.

Values of gravity determined from observations at a station are reduced to sea level, the mean surface of which is conceived as extended by canals under the continents and is called the geoid. In the 19th century, reduction to sea level was calculated by a simple formula which was derived by a permissible approximation from the law of gravitation. If h is the elevation of a station, the increment of gravity on reduction to sea level is $2gh/r$, where r is the radius of the earth and g is the intensity of gravity. The increment generally was diminished by subtracting the attraction of the mass of earth between a station and the extended sea level. A commonly used reduction was the Bouguer reduction. The factor in this reduction is such that if the density of the attracting mass is assumed to be one-half of the average density of the earth, the resultant reduction to sea level is $\frac{5}{8}$ of that which arises from reduction for elevation alone.

The calculation of the ellipticity is accomplished by means of a theorem which was set forth in the 18th century by Clairaut in *Figure de la Terre*, 1743. In its first approximation, Clairaut's theorem expresses ellipticity, e, by the equation,

$$e = 5/2\ c - b,$$

in which the symbol c designates the ratio of centrifugal force per unit mass at the equator to the absolute value of gravity at the equator, and b is a constant which occurs in the theoretical formula for gravity at sea level in latitude ϕ as a function of gravity at equatorial sea level, g_e,

$$g_0 = g_e\ (1 + b \sin^2 \phi).$$

The preceding equations to the first approximation for ellipticity and theoretical gravity were given in the treatise *Geodesy* by Colonel A. R. Clarke, as published in 1880. This work is referred to by Peirce on several occasions and undoubtedly influenced his work in the field. Subsequently, an expanded form of Clairaut's theorem was set forth by F. R. Helmert in *Die mathematischen und physikalischen Theorien der höheren Geodäsie*, Volume I (1880), Volume II 1884). Records of books which were used by Peirce list Helmert's work, and he refers to it in reports and fragments of manuscripts. As has already been noted, the formula for theoretical gravity contains the function $\sin^2 \phi$ of the second order. Helmert also worked with a function of the fourth order, the fourth power of the sine of the latitude. Peirce went further than Helmert, and in a published paper used a third order term. In a subsequent report to the Superintendent of the Coast and Geodetic Survey, he also proposed a longitude term and a fourth order term in the latitude. A study of Peirce's work, both published and unpublished, reveals his initiative, originality, nd competence in mathematical geodesy.

III

The first published record of Peirce's methods in mathematical geodesy is his brief paper, "Results of Pendulum Experiments," in the *American Journal of Science*, October, 1880.[2] In this paper, he reported values of gravity expressed as the length of the seconds pendulum at stations in Hoboken, Paris, Berlin, and Kew Observatory: (1) at the station; (2) as reduced to sea level; (3) as reduced to the equator at sea level. He gives no information concerning the elevation of these stations, but such information is obtainable from his memoir, "Measurements of gravity at initial stations in America and Europe,"[3] dated December 13, 1878. Helmert, in II, p. 209, analyzes Peirce's published results and calculates the elevations of the stations on the basis of his own information and on the assumption that the reduction for elevation and attraction is 0.74 of the free-air reduction. The present writer, from the information in Peirce's just cited memoir, and his description of the support for his pendulum apparatus at Kew,[4] has adopted elevations above sea level: for Hoboken, 10 meters; Paris, 74 meters; Berlin, 38 meters; Kew, 22 feet, and has calculated that Peirce's reduction to sea level is about 0.70 of the free-air reduction.

The reduction to the equator from station sea level was the second step in Peirce's report. From known latitudes of the stations, the squares of the sines of the latitudes are calculable. Application of the first approximation to the theoretical formula for gravity yields a value for the coefficient b which is concordant with the value 0.0052286 which Clarke derived by the method of least squares from English and Indian Surveys of gravity.

Peirce's first original contribution to mathematical geodesy is his previously cited memoir, "On the deduction of the ellipticity of the earth from pendulum experiments." Early in the 19th century, Captain Kater was the first to construct and use a convertible pendulum, one which could be oscillated, about two knife-edges, for the determination of absolute values of gravity. He also constructed a number of invariable pendulums which could be swung on one knife-edge only, and thus could be used solely for the determination of relative values. Relative values are referred to an initial station by a theorem that the ratio of values of gravity at two stations is equal to the ratio of the squares of the frequencies of an invariable pendulum which has been oscillated at the two stations. Such Kater pendulums were swung at stations throughout the world. In his memoir on ellipticity, Peirce gave the results of observations with Kater invariable pendulums, as corrected by him for temperature, pressure, and elevation. By the method of least squares, he derived from the various values of gravity the value 0.0052375 for the coefficient b in Clairaut's theorem. From the formula for ellipticity, in its first approximation, Peirce calculated the ellipticity as 1/291.5, and indicated the probable error in the denominator as ± 0.9. Peirce set up a formula for continental attraction in his memoir, but stated that the calculation was so laborious that he diminished the correction for elevation alone by one-tenth of its value as allowance for continental attraction in reductions to sea level. As previously noted, Peirce's value for the ellipticity was published by C. Wolf in a table of values from Newton to Helmert.

Major J. E. Herschel (later Lieutenant-Colonel), of the Royal Engineers, brought three Kater invariable pendulums to the United States from England in 1882, for the purpose of connecting American and European stations by determinations of relative gravity. After oscillating these pendulums at various stations, Major Herschel left the instruments in the custody of the Coast and Geodetic Survey. The pendulums were used by Edwin Smith on a survey in the South Pacific in 1883, and his results provided Peirce with additional material for his calculations.

IV

Peirce wrote to the Superintendent, June 30, 1888,[5] that he felt he ought to make a new calculation of the "compression of the earth." For this new calculation, he reported that he had used 89 stations occupied with Kater pendulums, embracing those of Herschel and Smith. Peirce wrote, "Besides the usual term depending on the sine of the latitude I have introduced another with the factor $\sin^3\phi - 3/5 \sin\phi$, which diminishes the residuals decidedly. I have also determined the coefficient of the correction for elevation from the observations themselves by least squares. I have corrected an error which I previously fell into (in common, I believe, with all those who have deduced the figure of the earth from pendulum observations), in that instead of taking for 2/5 the sum of the compression and the coefficient of latitude correction of gravity, the ratio of centrifugal force at the equator to the value of gravity at the equator, I now take the ratio of centrifugal force at the equator to the mean value of gravity. I obtain as the value of the compression 1/291.6. The coefficient of $(\sin^3\phi - 3/5 \sin\phi)$ is 1.21 s."

In a letter to Professor A. M. Mayer, May 25, 1887,[6] Peirce gave the value of gravity at the equator as 978.1 cm./sec.^{2}, and at the pole as 983.2 cm./sec.^{2}. Substituting the new compression and the average value of gravity in Clairaut's theorem yields 0.0052161 as the value of the coefficient b in the formula for theoretical gravity.

Peirce's letter continued, "The mean absolute acceleration of gravity which is required in deducing the figure of the earth has been calculated from my determination at Paris, Kew and Washington. I ought to include also Hoboken, but my latest determination there is not reduced. My other stations are not used because they are not among those whose station errors are determined by the Kater pendulums. My Paris and Washington determinations give almost identical values for the mean gravity. The Kew observations give a value differing some 4 s. This is partly owing to the influence of a connection between Greenwich and Kew made through Madras, and partly to my ratio differing over 1 s. from Herschel. But there is a suspicion of error in the calculation which I have not yet investigated. Besides this, I have done some analytical work towards improving the 'fluid theory' of the earth's figure. This theory, rightly understood, does not imply that the earth ever was literally a fluid (however probable that may be on other grounds), but only that its figure yields in the course of ages to the stupendous forces considered, in regard to which, when indefinite time is allowed, the rigidity of any materials is as nothing. So understood and for the purpose of the figure of the earth, the theory appears to me hardly open to any doubt. True, the matter toward the center of the earth may not yield to the process, but that has little to do with the external figure."

The new procedures and coefficients to which Peirce refers in his letter of June 30, 1888 were utilized by him in the reduction of the observations which were made by Lieutenant Greely and Sergeant Israel at Fort Conger, Lady Franklin Bay, Grinnell Land, during the expedition to the Arctic region in the years 1882-83. A Peirce pendulum, No. 1, had been assigned to the expedition for this purpose, was swung at the Smithsonian as the initial station, was swung at Fort Conger, and after return to Washington under tragic circumstances, was again swung at the Smithsonian. The data were submitted to Peirce for reduction. He submitted a report in April, 1887, but revised it in proof during the following year. In a letter[7] to the Superintendent, July 31, 1888, Peirce wrote, "General Greely having sent me the proofs of my report on his pendulum experiments, which you will remember, I made contrary to my own judgment at a time when I was not yet ready to undertake it, I was obliged to do most of the work over again and rewrite the report. This time I brought it to a final result. This was sent to General Greely, and another set of proofs sent me, corrected and returned." Peirce's report on gravity at Fort Conger was published as an appendix to Volume II, *Report on the Expedition to Lady Franklin Bay*, by A. W. Greely, Washington, 1889. Peirce declared in his report that the new third order term was especially suitable for the Arctic station. The third order term occurs in a formula for gravity derived by G. W. Hill in 1884,[8] but

Peirce appears to have been the first to apply it in practice for geodesy. In 1901, Professor Helmert reported[9] that A. Ivanof had introduced this term in order to take account of the inequality between northern and southern hemisphere, a consideration which parallels Peirce's choice of the term for the Arctic station. Helmert also considered use of the third order term in the paper just cited, and subsequently considered it for a general gravity formula in 1915.[10]

V

Peirce reports in the letter of July 31, 1888, to which we have referred, his most innovative work in mathematical geodesy. He writes: "I have already reported that I calculated a small harmonic term of the third order and applied it to the variation of gravity. In my paper of 1881, the Indian work was connected with the rest solely by means of a single expedition which carried a single pendulum from London to Madras without returning it. But Herschel having connected Kew with London, a new mode of connection was given. I therefore worked out by least squares, a second formula for the distribution of gravity, introducing all Herschel's and Smith's work. But not being satisfied with this mode of treatment, I made a third calculation in which the expeditions were only connected at initial stations. The principal affect of this was to connect the Indian work with the rest by means of Herschel's work alone. No doubt this is the best way, though it increases the residuals considerably.

"This third calculation is not yet completed, since I propose to introduce a term depending on the fourth power of the latitude and another having for the variable factor,

$2 \sin^2 i \sin \phi - 2 \sin i \cos i \cos \phi \cos \psi + \sin \phi - 5 \sin^2 i \sin^3 \phi + 10 \sin i \cos i \sin^2 \phi \cos \phi \cos \psi + 5 \cos^2 i \sin \phi \cos^2 \phi \cos^2 \psi$, where i is the obliquity of the ecliptic, ϕ is the latitude, and ψ is the longitude from Greenwich. The values of these two terms have been already calculated in duplicate for all stations. The labor of applying them remains. The above term of the third order is something like what is wanted. It is probably susceptible of improvement.

"The mean figure of the earth is at present connected with the variation of gravity by means of Clairaut's formula. This formula I have corrected by carrying the calculation to another degree of approximation. The work has been exceedingly laborious, because owing to Mr. Risteen's having accepted another situation, I have had to repeat the whole work twice to insure accuracy. My paper on this subject is now getting copied for transmission to you for publication. The correction to the formula is important, since it is shown that the denominator of the compression as deduced by Clairaut's formula ought to be increased by two units. I think this discovery will increase materially the confidence of geodesists in the method of deducing the compression. I have incidentally carried to a second degree of approximation the formula for the correction for elevation, but the result differed by about 1 per cent from the formula I have been using."

On August 10, 1888, the Superintendent received from Peirce the manuscript of a new memoir on ellipticity, "On the Mean Figure of the Earth from determinations of Gravity, Second Paper." It was submitted to Assistant Charles A. Schott for comment. The latter rendered a noncommittal judgment on December 26, 1888.[7] It would appear that in this new paper Peirce discussed a theorem of Stokes, for Schott noted a correspondence between the work of Peirce on Stokes and that of Helmert ("whose work came under the author's notice while writing his report"). Schott reported that the numerical work in Peirce's paper fell back on his work in 1881.

Peirce's modification of Clairaut's theorem is not determinable, since documents supplementing the foregoing quotation are not available. It is pertinent to the present discussion, however, to comment upon his reported increase by two units of the denominator in the compression, therefore a decrease in the ellipticity. The change may have resulted from an increase in the coefficient b, or from new terms in Clairaut's theorem, or from both. In a subsequent paper, presently to be introduced, and in fragments of manuscripts dated as subsequent to 1888, Peirce used the value 0.0052375 for the coefficient b, so that this earlier value of this coefficient was restored with the modification of Clairaut's theorem. Peirce's second paper on the figure of the earth was withdrawn from publication, for the reason given by Peirce that it incorporated unpublished results by Herschel. In view of the unavailability of this paper, and of documents on Peirce's modification of Clairaut's theorem, one is not able to compare the innovative work of two experts in mathematical geodesy, Director F. R. Helmert of the Royal Prussian Geodetic Institute, and Charles S. Peirce of the United States Coast and Geodetic Survey.

Peirce's innovative third term was published in his report on gravity at Fort Conger. We face, however, the further problem: Did Peirce use the new latitude and longitude terms in the theoretical formula for gravity?

VI

In the course of his research on gravity, Peirce designed and had constructed examples of a new type of pendulum, an invariable, reversible pendulum with two knife-edges and of symmetrical, cylindrical form which rendered possible theoretical calculations of the effect of the viscosity of the medium upon oscillations. With Peirce Pendulums Nos. 1, 2, 3, 4, he made observations for the measurement of gravity at various stations, notably at the Smithsonian Institution, Washington, D. C.; at the University of Michigan, Ann Arbor; at the University of Wisconsin, Madison; and at Cornell University, Ithaca; during the period 1884-86. In August, 1886, he was relieved of duties in the field to prepare results of observations for publication. In the subsequent period, he also carried on the theoretical researches as already reported.

On November 20, 1889, Peirce submitted the manuscript of a *Report on Gravity at the Smithsonian, Ann Arbor, Madison, and Cornell.* The *Report* was denied publication and remained unlisted and unknown until it was found in January, 1968, and copies were made available through the courtesy of Admiral Tison, Director of the Coast and Geodetic Survey. The present writer has reported on Peirce's recovered *Report* in "An Unpublised Scientific Monograph by C. S. Peirce."[11] The principal section of the *Report* presents results of observations, corrections to observations, and tables of coefficients for the reduction of observed gravity to sea level and the equator, for the determination of relative values of gravity at the four stations specified in the title. Peirce gives the value 0.0052375 for the coefficient b of the square of the sine of the latitude in the theoretical formula for gravity. The reduction to sea level is for height alone, the free-air reduction, and he gives the value of the coefficient of the height as 0.004156 logarithmic seconds per foot. For the reduction of observed gravity to the equator, Peirce does not give coefficients for the third order term, a fourth power term, and the longitude term, which were mentioned in his report to the Superintendent on the third calculation for the figure of the earth.

A short section, "On the Absolute Value of Gravity," follows the main section of Peirce's *Report* of 1889. There is presented a table of absolute values of equatorial gravity, expressed as the lengths of the seconds pendulum in centimeters, as determined from observations with the Repsold apparatus at stations in Hoboken, Paris, and Kew Observatory, during the years 1875-77:

Hoboken	99.09531
Paris	99.09931
Kew	99.09442

There follows a table of results of observations and steps in the process of obtaining equatorial values by application of transformation and station errors. The values of the transformation functions, after reductions to sea level, are markedly greater than those which result from calculations with the coefficient $b = 0.0052375$ alone. The greater values are to be expected, in view of the decrease in the equatorial value of gravity from earlier values used by Peirce. Peirce's sole use of $b = 0.0052375$ for the transformation in 1881 was correlated with the value 991 millimeters as the length of the equatorial seconds pendulum. The value 99.095 cms. for equatorial gravity in the 1889 *Report* calls for supplementation of the transformation, as based solely on b, by the addition of new terms. It has been found if one uses the previously cited value 0.0052375 for b, adds the third order term with the coefficient 1.21 log. secs., adds the fourth power term assumed to be the spherical harmonic $\sin^2 \phi \cos^2 \phi$, and adds the longitude term previously cited, one can determine coefficients for the fourth power and longitude terms which yield the corrected given values of the transformation functions for Hoboken, Paris, and Kew.

Peirce gives in logarithmic form the values of the functions for transformation from equatorial sea level gravity to gravity at a station above sea level in a specific latitude and longitude, as follows:

Hoboken	0.0009761
Paris	0.0012891
Kew	0.0014007

Now, as we have previously seen, the first approximation to the theoretical formula for gravity is

$$g_o = g_e\ (1 + b \sin^2 \phi).$$

The complete formula presumably used by Peirce in his *Report* of 1889 is

$$g = g_e\ (1 + bB + cC + dD + eE - ah),$$

where $B = \sin^2 \phi$, $C = \sin^3 \phi - 3/5 \sin \phi$, $D = \sin^2 \phi \cos^2 \phi$, E is the previously given longitude term, and h is the elevation above sea level; b, c, d, e, and a are respective coefficients. The expressions in the parentheses are the transformation functions, and the logarithms of the extended functions for Hoboken, Paris, and Kew are assumed to be given at the opening of the present paragraph.

The coefficient of elevation h is derivable from the value, 0.004156 log. secs. per foot; it is 0.00000009359 for h in feet. The coefficient of the third order term C as derivable from 1.21 log. sec. is 0.0000279. The coefficient b of the second order term is assumed to be 0.0052375, the value which Peirce uses in the 1889 *Report.* This leaves as unknowns the coefficients d and e, of the fourth order term D and the longitude term E, respectively. An initial confirmation of the choice of the coefficient b is that on treating b, d, and e as three unknowns which are determined by three simultaneous equations, the solution for b yields the value 0.0052375 precisely. The additional constants are found to be

$$d = 0.00009927$$
$$e = 0.00000129.$$

The results just presented are based upon the *Report* of 1889, in which Peirce gave the lengths of the equatorial seconds pendulum, as determined from observations at Hoboken, Paris, and Kew. In a letter to the Superintendent, July 3, 1890,[12] he presented new values for those stations, respectively, 99.09537, 99.09940, and 99.09445. If we assume that Peirce used the same station errors and elevation corrections as in the *Report* of 1889, the resulting values of the transformations give slightly lower values for the constants, and a discrepancy of one unit in the seventh decimal place in the check, than do the earlier values. Since the presuppositions of the later calculation are uncertain, we shall retain the previously cited coefficients for the new latitude and longitude terms in Peirce's theoretical formula for gravity.

VII

Peirce's letter of July 3, 1890, to which we have just referred, also provides material for a possible independent confirmation of the preceding results. He states, "The value assumed for the mean equatorial seconds' pendulum is also in accord with that derived by a similar process from the determinations with Bessel's apparatus at Koenigsberg and Berlin, these stations being reduced to Altona by means of Peters' work with the Lohmeier station. Namely, Bessel's determinations so reduced give the following values of the mean equatorial seconds' pendulum.

From the Berlin determinations	99.09719 cms.
Koenigsberg	99.09298
Mean	99.09508."

Peirce used values of gravity as reduced to Altona, undoubtedly because he had a value of the station-error for that station. In his 1881 memoir on the ellipticity of the earth, he gave this station-error in logarithmic terms as +0.0000013, a relatively small value.

Bessel's results, which he expressed in terms of Parisian Lines, were given by Peters in his contributions to *Astronomische Nachrichten*,[13] and also by Helmert in II, p. 209: 440,7390 lines for Berlin, and 440,8179 lines for Koenigsberg, of the "Toise du Perou," a copy of which served Bessel as his standard of length. Both Peters and Helmert also gave Bessel's results in meters, as based upon the French legal definition of the meter as 443,296 lines. But these values in turn required correction in view of new determinations of Bessel's unit. Helmert, II, p. 210, states that the difference of Peirce with Bessel for Berlin is to be attributed in part to the circumstance that the meter was then defined some 0.008 mm. smaller than legally by the toise. Helmert gave a correction of +13 microns to Bessel's values in meters; this is in conformity with the report of Dr. Benoit, of the International Bureau of Weights and Measures, that Bessel's toise was 26 microns shorter than the originally accepted 1.949.0348 millimeters.[14]

Now, Peters' work with the Lohmeier reversible pendulum indicated that the length of the seconds pendulum at Altona was 114.3 microns greater than at Berlin, and 103.1 microns less than at Koenigsberg. Upon applying these differences to the corrected Bessel's values, one finds that the length of the seconds pendulum at Altona, at sea level, is 99.4362 cms., as reduced from Berlin, and 99.4320 cms., as reduced from Koenigsberg. If we adopt Peirce's equatorial values as derived by him from Bessel's work at Berlin and Koenigsberg, respectively, then apply the transformation from equator to Altona, and also introduce the 1881 station-error for Altona, Peirce's values come out to be two microns less than the above cited values from Bessels and Peters in each case. This is fairly good agreement, in view of the circumstance that the two values for Altona differ by 42 microns.

The foregoing comparison, however, is based upon Peirce's values of 1878-80, as corrected by him by the subtraction of 16.2 microns, and reported in a note to a paper which he had read to the Academy of Sciences, Paris, June 14, 1880.[15] The correction was based, in part, upon a report concerning Peirce's meter No. 49, from Professor Foerster, of the Imperial Standards Commission, Berlin. Helmert used Peirce's values of 1878-1880, recognized uncertainties in Peirce's standards, and gave a correction of -13 microns to the latter's values. If we now accept the correction to Peirce by Helmert, we add 3 microns to Peirce's values and thus bring his values for Altona 1 micron higher than those derived from Bessel's work. The uncertainties in the comparison originate in lack of knowledge of Peirce's later station-error for Altona and of corrections which he used in order to make his results comparable with those of Bessel. Nevertheless, in view of discrepancies in the determinations of gravity by different observers, the relative closeness of values for Altona, based on calculations from Bessel and Peirce, respectively, furnishes evidence for the thesis of this paper, namely, that Peirce used a formula for theoretical gravity which contains original latitude and longitudinal terms.

The introduction of a third order term, a new fourth order term, and a longitude term in the theoretical formula for gravity does account for the values of the transformation functions for Hoboken, Paris and Kew in Peirce's *Report on Gravity* of 1889. This conclusion appears to be inconsistent with the sole use of the principal term, the square of the sine of the latitude, in the reductions to equatorial gravity of relative gravity at the Smithsonian and related stations. A difference in the treatment of absolute and relative values of gravity is characteristic of Peirce. His lack of use of the new terms in reductions for the American stations may arise from the circumstance that the totals of the supplementary terms for these stations are practically equal. For the Smithsonian, the station-error may have been adjusted, inasmuch as the station-error of +1.90 log secs. is an appropriate increase from the value +0.76 log. secs. of the *Report* on Fort Conger.

In the present discussion of Peirce's Report on Gravity of 1889, attention has been directed to the theoretical formula for gravity. One should also take notice, however, of the result to which Peirce gave the first place in his *Report*. Now, a principal problem of mathematical geodesy is the determination of the form of the geoid, which has been introduced earlier in this paper as the mean surface of the sea imagined as extended under the continents by canals. A solution of this problem is in principle obtainable from a formula which was set forth by Stokes in 1849 in a paper "On the Variation of Gravity at the Surface of the Earth."[16] Stokes' formula expresses the distance of the geoid from the spheroid of reference in terms of an integral of anomalies of gravities extended over the surface of the earth. In his *Report* of 1889, Peirce gives the first statement of results under the heading, "Form of the Geoid according to the Observations." He states, "Though the data are yet too scanty to enable us to calculate the form of the geoid, yet we can from the observations assign

values which the departure from the mean spheroid cannot possibly exceed." From Stokes' theory, he derived a formula for provisional maximum values of the height of the geoid above the mean spheroid from station-errors,

$$\Delta r \text{ (in feet)} \leq (1/0.00069)\ \Delta N,$$

and calculated values given in Table I

Smithsonian	+ 275 feet
Ann Arbor	+ 190
Madison	+ 123
Cornell	— 86

This was a provisional, but pioneer, contribution to the general problem of the determination of the geoid.

VIII

With the failure of publication of his *Report on Gravity* of 1889, and his subsequent severance from the Survey, Peirce was not able to win recognition for his original initiatives in mathematical geodesy. The third order harmonic, and also longitude terms, had been introduced in a formula for gravity with 20 constants in 1884 by G. W. Hill, but geodesists do not appear to have used Hill's elaborate formula. Peirce did use the third order term in the published report on Fort Conger. This term subsequently was used by A. Ivanof in 1898, and it was considered by Helmert in papers of 1901 and 1915. In the latter paper, Helmert also introduced a longitude term, and this procedure was adopted by other geodesists. The contemporary International Gravity Formula contains two terms, a principal term in the square of the sine of the latitude, and a term in the square the sine of twice the latitude. The latter function corresponds to the product of the squares of the sines and cosines of the latitude which we have assumed for Peirce. The value of Peirce's longitude term, which differs in structure from that introduced by Helmert and others, was not subjected to the test of publication.

The present paper has given an account of Peirce's work as mathematical geodesist. His calculation of the ellipticity of the earth in 1881 has a place in the history of such determinations. He introduced new terms in the theoretical formula for gravity, and worked at an extension of Clairaut's theorem. He utilized Stokes' theory for the form of the geoid and made a pioneer calculation for American stations.

While he was not able to continue his projects by reason of severance from the Coast and Geodetic Survey, December 31, 1891, the record demonstrates that during his service Peirce maintained his interest, held to his standards of excellence, and envisioned future developments in mathematical geodesy.

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APPENDIX

Data	Hoboken	Paris	Kew	Altona
Latitude North	40° 44′ 30″	48° 50′ 14″	51° 28′ 06″	53° 32′ 45″
Longitude from Greenwich	-74° 02′	2° 20′ 12″	-0° 19′	9° 56′
Log. Trans. Equat-Stat.	0.0009761	0.0012891	0.0014007	
Elevation	10 meters	74. m.	22 feet	31 m.
Coeff. of h in feet, a	0.00000009569			
Coeff. b	0.0052375			
Coeff. c	0.0000279			
$B = \sin^2 \phi$				
$C = \sin^3 \phi - 3/5 \sin \phi$				
$D = \sin^2 \phi \cos^2 \phi$				
E = longitude term				
Determined				
Coeff. d = 0.00009929				
Coeff. e = 0.00000129				
Calculated				
bB	0.0022309 2	0.0029684 7	0.0032050 2	0.0033884 8
cC	-0.0000031 7	-0.0000007 0	0.0000002 6	0.0000010 6
dD	0.0000242 8	0.0000243 8	0.0000235 8	0.0000226 8
eE	0.0000012 0	0.0000037 6	0.0000037 1	0.0000035 7
ah	0.0000031 4	0.0000231 6	0.0000021 0	0.0000000 0
Transformation Equator to Station: (1 + bB + cC + dD + eE - ah)				
	1.0022500 9	1.0029727 5	1.0032304 7	1.0034157 9
Log. of Transformation	0.0009761	0.0012891	0.0014007	

Altona: Station-error (1881) + 0.0000030, to be added to transformation

Length of seconds pendulum at Altona, at sea level:

From Berlin		By Bessel and Peters	99.4362 cms.
		Peirce	99.4360
	Koenigsberg	Bessel and Peters	99.4320 cms.
		Peirce	99.4318.

NOTES

1. *U. S. Coast and Geodetic Survey. Report of the Superintendent, June, 1881,* Appendix No. 15. Washington, 1883.
2. Volume XX, p. 327.
3. *Report of the Superintendent, June, 1876,* Appendix No. 15. Washington, 1879.

4. *U. S. Coast and Geodetic Survey. Assistants' Reports, L-Q, 1883.* Letter dated August 1, 1883.
5. *Assistants' Reports, N-R, 1888.*
6. I am indebted to Professor Max H. Fisch for this reference. The date 1887 given for the letter corrects an error in "An Unpublished Scientific Monograph by C. S. Peirce," *Trans. Charles S. Peirce Soc.*, Winter 1969/Vol. V. No. 1, p. 19.
7. Note. 5.
8. George William Hill, *The Collected Mathematical Works,* II, p. 315. Carnegie Institution of Washington, 1906.
9. *Sitzb. d. Preuss. Akad. d. Wiss., math.-natur. Kl.,* 14, 1901.
10. Ibid., 41, 1915.
11. op. cit., n. 6.
12. *Assistants' Reports, J-Q, 1890.*
13. *Astr. Nach.*, Vol. 97, 1; Vol. 98, 65, 99; Vol. 99, 129, 380. 1880-1881.
14. C. R. Deuxième Conférence Générale des Poids et Mesures, Paris, 1895. *Travaux et Mémoires du Bureau International des Poids et Mesures,* Tome XII, pp. 42-44, 1902.
15. "On the Value of Gravity at Paris." *Report of the Superintendent. June, 1881,* Appendix No. 17, p. 463. Washington, 1883.
16. *Trans. Cambridge Philosophical Society*, Vol. 8, 1849.