APPENDIX No. 16.

ON A METHOD OF OBSERVING THE COINCIDENCE OF VIBRATION OF TWO PENDULUMS.

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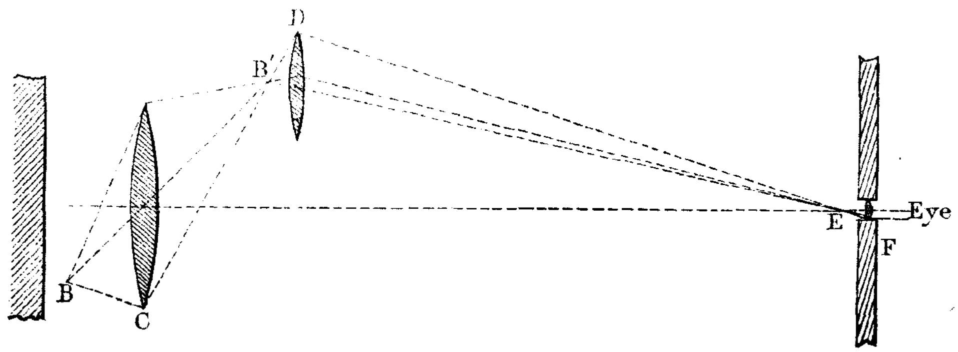
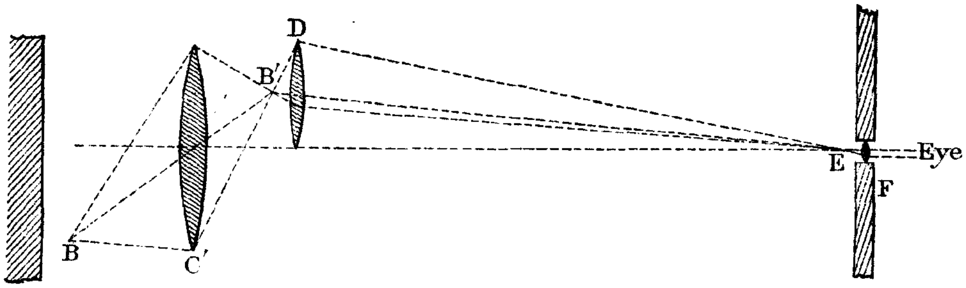
DEAR SIR: I have made a full set of experiments with different methods of observing the coincidences of two pendulums. By far the most accurate method is the following:





To the wall A of a small chamber is fixed a clock which carries on its pendulum a brilliantly illuminated horizontal scale, say of half millimeters. B represents the middle point of this scale. C or C′ is a large achromatic lens placed so that an image of scale will be formed at B′ at a fixed distance from the wall. There are two positions, C and C′, which the lens may have to effect this. In one position the amplitude of vibration of B is multiplied in a certain ratio, say *r;* in the other position it is diminished in the same ratio. This is a well-known optical principle. The lens moves in a slide, by means of strings, and up to stops, so that it can be drawn at any time from one of these positions to the other. At D D is the plane of oscillation of the pendulum on knife-edges, which measures the force of gravity. The plane of motion of D is parallel to that of B. This pendulum carries a lens which brings the image at B′ at focus at E close to the opposite wall F of the room.• When the amplitude of D is to the amplitude of B′ as E D is to E B′, the image remains stationary at E, provided the pendulums are in coincidence. The image E is observed by means of an eye-piece, G, fixed in the wall.

The effect is this: The lens C being in the position C (the nearer to B), and the pendulum D oscillating at nearly the right amplitude, the image of the scale will generally flash across the field of vision so rapidly that it can only be seen at the instant of reversing its direction. But as the pendulums approach coincidence it moves less and less, and if the two amplitudes are precisely in the right proportion it finally comes absolutely to rest with the middle of the scale just on the cross-wire of the eye-piece (*i. e.*, just where it would be with the pendulums both at rest). As a general rule, however, it does not come absolutely to rest, but finally gets over, say a millimeter in a second, after which it begins to move faster. The approach to and departure from the minimum amplitude is not very gradual but rather sudden, so that there is no difficulty at all in deciding which is the minimum oscillation. The observer has to note at what second the minimum oscillation occurs, and also what part of the scale is on the cross-wire at the turning points before and after this oscillation; then, by the application of a formula, the time can readily be determined to near $\frac{1}{1000}$th of a second. The lens C is then pulled forward to the position C′, and the observation is repeated when the pendulum has diminished its arc of oscillation sufficiently.

The formula which applies is as follows: Let s be the apparent oscillation of the scale; then,

$$s=a_1 \cos (b_1t+c_1)-a_2 \cos (b_2t+c_2)$$

$$=-(a_1+a_2) \sin \left(\frac{b_1+b_2}{2} t+\frac{c_1+c_2}{2}\right) \sin \left(\frac{b_1-b_2}{2} t+\frac{c_1-c_2}{2}\right)$$

$$+(a_1-a_2) \cos \left(\frac{b_1+b_2}{2} t+\frac{c_1+c_2}{2}\right) \cos \left(\frac{b_1-b_2}{2} t+\frac{c_1-c_2}{2}\right)$$

Here a_1 and a_2 are the amplitudes of the knife-edge pendulum and of the image of the other formed at B′ and reduced in the ratio $\frac{\mathrm{ED}}{\mathrm{EB}'}$. b_1 and b_2 are the reciprocals of the periods of the two oscillations multiplied by 180°; c_1 and c_2 depend upon the initial conditions. Since b_1 and b_2 differ very little in value (about $\frac{1}{150}$), it follows that sin and cos $\left(\frac{b_1+b_2}{2} t+\frac{c_1+c_2}{2}\right)$ go through all their values in about two seconds, while sin and cos $\left(\frac{b_1-b_2}{2} t+\frac{c_1-c_2}{2}\right)$ go through their values in about five minutes. We thus see why the scale should appear to oscillate back and forward in a second with a changing amplitude. If a_1 did not change, the amplitude would go through its cycle of changes in five minutes.

Let us see what the amplitude of oscillation of s is for a particular value of $\frac{b_1-b_2}{2} t+\frac{c_1-c_2}{2}$. Considering this as fixed, the turning takes place when

$$(a_1+a_2) \cos \left(\frac{b_1+b_2}{2}+\frac{c_1+c_2}{2}\right) \sin \left(\frac{b_1-b_2}{2} t+\frac{c_1-c_2}{2}\right)$$

$$+(a_1-a_2) \sin \left(\frac{b_1+b_2}{2}+\frac{c_1+c_2}{2}\right) \cos \left(\frac{b_1-b_2}{2} t+\frac{c_1-c_2}{2}\right)=0$$

Or when

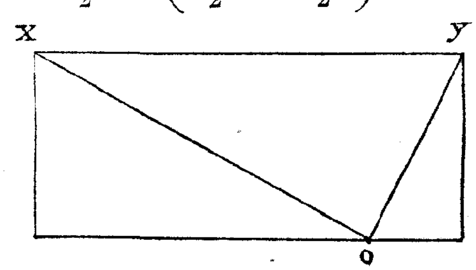
$$-\frac{a_1+a_2}{a_1-a_2} \tan \left(\frac{b_1-b_2}{2} t+\frac{c_1-c_2}{2}\right)=\tan \left(\frac{b_1+b_2}{2} t+\frac{c_1+c_2}{2}\right)$$

Putting in the figure below,

$$\frac{a_1+a_2}{2} \sin \left(\frac{b_1-b_2}{2} t+\frac{c_1-c_2}{2}\right)=\mathrm{OX}$$

and

$$\frac{a_1-a_2}{2} \cos \left(\frac{b_1-b_2}{2} t+\frac{c_1-c_2}{2}\right)=\mathrm{OY}$$



we see that the maximum value of s, that is, the value at the turning point, is

$$\sqrt{(\mathrm{OX})^2+(\mathrm{OY})^2}$$

which is

$$\sqrt{(a_1^2+2a_1a_2+a_2^2)\sin^2\left(\frac{b_1-b_2}{2}t+\frac{c_1-c_2}{2}\right)+(a_1^2-2a_1a_2+a_2^2)\cos^2\left(\frac{b_1-b_2}{2}t+\frac{c_1-c_2}{2}\right)}$$

$$=\sqrt{a_1^2+a_2^2-2a_1a_2\cos\left((b_1-b_2)\,t+c_1-c_2\right)}$$

This is the amplitude of the apparent oscillation of the scale. Its greatest value is a_1+a_2 and its least value is a_1-a_2. A little calculation will show that supposing a_1 to be one twenty-fifth part larger than a_2, the oscillation next to the smallest has double the amplitude of the smallest. If, therefore, we only sought to know the coincidence within one second (giving the time to $\frac{1}{150}$th of a second) no calculation would be necessary; but we can find the time of coincidence nearer than a second.

For this purpose we require the precise condition which defines the moment of turning. It is

$$\left(-(a_1+a_2)\frac{b_1+b_2}{2}-(a_1-a_2)\frac{b_1-b_2}{2}\right)\cos\left(\frac{b_1+b_2}{2}t+\frac{c_1+c_2}{2}\right)\sin\left(\frac{b_1-b_2}{2}t+\frac{c_1-c_2}{2}\right)$$

$$+\left(-(a_1+a_2)\frac{b_1-b_2}{2}-(a_1-a_2)\frac{b_1+b_2}{2}\right)\sin\left(\frac{b_1+b_2}{2}t+\frac{c_1+c_2}{2}\right)\cos\left(\frac{b_1-b_2}{2}t+\frac{c_1-c_2}{2}\right)$$

$$=-(a_1b_1+a_2b_2)\cos\left(\frac{b_1+b_2}{2}t+\frac{c_1+c_2}{2}\right)\sin\left(\frac{b_1-b_2}{2}t+\frac{c_1-c_2}{2}\right)$$

$$-(a_1b_1-a_2b_2)\sin\left(\frac{b_1+b_2}{2}t+\frac{c_1+c_2}{2}\right)\cos\left(\frac{b_1-b_2}{2}t+\frac{c_1-c_2}{2}\right)=0.$$

Hence

$$\tan\left(\frac{b_1+b_2}{2}t+\frac{c_1+c_2}{2}\right)=-\frac{a_1b_1+a_2b_2}{a_1b_1-a_2b_2}\tan\left(\frac{b_1-b_2}{2}t+\frac{c_1-c_2}{2}\right)$$

$$\sin\left(\frac{b_1+b_2}{2}t+\frac{c_1+c_2}{2}\right)=\frac{-\frac{a_1b_1+a_2b_2}{a_1b_1-a_2b_2}\tan\left(\frac{b_1-b_2}{2}t+\frac{c_1-c_2}{2}\right)}{\sqrt{1+\left(\frac{a_1b_1+a_2b_2}{a_1b_1-a_2b_2}\right)^2\tan^2\left(\frac{b_1-b_2}{2}t+\frac{c_1-c_2}{2}\right)}}$$

$$=\frac{\mp(a_1b_1+a_2b_2)\sin\left(\frac{b_1-b_2}{2}t+\frac{c_1-c_2}{2}\right)}{\sqrt{(a_1b_1-a_2b_2)^2\cos^2\left(\frac{b_1-b_2}{2}t+\frac{c_1-c_2}{2}\right)+(a_1b_1+a_2b_2)^2\sin^2\left(\frac{b_1-b_2}{2}t+\frac{c_1-c_2}{2}\right)}}$$

$$\cos\left(\frac{b_1+b_2}{2}t+\frac{c_1+c_2}{2}\right)=\frac{\pm(a_1b_1-a_1b_2)\cos\left(\frac{b_1-b_2}{2}t+\frac{c_1-c_2}{2}\right)}{\sqrt{(a_1b_1-a_2b_2)^2\cos^2\left(\frac{b_1-b_2}{2}t+\frac{c_1-c_2}{2}\right)+(a_1b_1+a_2b_2)^2\sin^2\left(\frac{b_1-b_2}{2}t+\frac{c_1-c_2}{2}\right)}}$$

Hence

$$s=\pm\frac{(a_1+a_2)(a_1b_1+a_2b_2)\sin^2\left(\frac{b_1-b_2}{2}t+\frac{c_1-c_2}{2}\right)\mp(a_1-a_2)(a_1b_1-a_2b_2)\cos^2\left(\frac{b_1-b_2}{2}t+\frac{c_1-c_2}{2}\right)}{\sqrt{(a_1b_1+a_2b_2)^2\sin^2\left(\frac{b_1-b_2}{2}t+\frac{c_1-c_2}{2}\right)+(a_1b_1-a_2b_2)^2\cos^2\left(\frac{b_1-b_2}{2}t+\frac{c_1-c_2}{2}\right)}}$$

The coincidence occurs when

$$\frac{b_1-b_2}{2}t+\frac{c_1-c_2}{2}=0.$$

When the turning is near coincidence, so that this is a very small quantity,

$$\frac{b_1-b_2}{2}t+\frac{c_1-c_2}{2}=\frac{b_1-b_2}{2}dt$$

$$s=\pm\frac{(a_1-a_2)(a_1b_1-a_2b_2)+\left\{(a_1+a_2)(a_1b_1+a_2b_2)-(a_1-a_2)(a_1b_1-a_2b_2)\right\}\frac{(b_1-b_2)^2}{4}(dt)^2}{\sqrt{(ab_1-a_2b_2)^2+\left\{(a_1b_1+a_2b_2)^2-(a_1b_1-a_2b_2)^2\right\}\frac{(b_1-b_2)^2}{4}(dt)^2}}$$

$$=\pm\frac{(a_1-a_2)+\frac{1}{2}\frac{a_1a_2(b_1+b_2)}{a_1b_1-a_2b_2}(b_1-b_2)^2(dt)^2}{\sqrt{1+\frac{a_1a_2b_1b_2}{(a_1b_1-a_2b_2)^2}(b_1-b_2)^2(dt)^2}}$$

$$=\pm\left\{a_1-a_2)+\frac{a_1a_2(a_1b_1^2-a_2b_2^2)(b_1-b_2)^2}{(a_1b_1-a_2b_2)^2}(dt)^2\right\}$$

By observing the value of s on the turning points of the smallest oscillation the amplitude will give (a_1-a_2), and the difference of amplitudes on the two sides will give dt to about a sixth of a second on substituting in the last equation the known values of a_2b_1 and b_2 and the value of a_1 determined from the amplitude. This will determine the time to about $\frac{1}{1000}$ of a second.

My opinion, however, is, that the best way of making pendulum observations is with my relay described in my printed paper.

Yours, very respectfully,

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Superintendent Coast and Geodetic Survey.

NOTE.

Since witnessing Major Herschel's experiments, I have done some additional work with the method of coincidences. I have used a scale of half millimeters pasted on the clock pendulum, and brought to focus by a good lens, on the plane of oscillation of the point of a fine cambric needle placed vertically on the gravity pendulum. The correction for decrement of arc—an effect Major Herschel detected and for which Mr. Farquhar has obtained a formula—is considerable in the case of the reversible pendulums. I have read off its value from a diagram constructed for the purpose.

FEBRUARY 20, 1883.