## Appendix No. 16.

NOTE ON THE EFFECT OF THE FLEXURE OF A PENDULUM UPON ITS PERIOD OF OSCILLATION.

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In determining the acceleration of gravity, it is necessary to consider the effect of the flexure of the pendulum during its oscillation.

Suppose, first, that a pendulum, otherwise rigid, has an elastic joint parallel to the knife-edge and at a distance $r$ vertically below it in the position of repose. Let $m_0$ be the mass of the piece of the pendulum connected with the knife, $m$ that of the piece jointed to it; $h_0$ the distance of the center of mass of the first piece below the knife-edge in the position of repose, $h$ that of the second piece below the joint; $\gamma_0$ and $\gamma$ the radii of gyration of the two pieces about axes through their centers of mass parallel to the knife-edge; $\theta_0$ and $\theta=\theta_0+\delta\theta$ the angular displacements of the two pieces about their centers from the position of repose of the whole pendulum. Let $\varepsilon$ be the coefficient of elasticity of the joint. With this notation, the kinetic potency* is written

$$U=gm_0 h_0 (1-\cos \theta_0)+gmr (1-\cos \theta_0)+gmh (1-\cos \theta)+\tfrac{1}{2}\varepsilon (\delta\theta)^2$$

If the differential equations were formed, were made linear by the omission of terms of higher degrees, and were then resolved, the motion would be seen to consist of two harmonic components, the amplitudes of which are arbitrary constants. It is, therefore, possible for one of these to vanish, and we may assume such an equation between $\theta$ and $\theta_0$ as to bring about this result. This equation must express the condition that the kinetic potency shall be a minimum. It is, therefore,

$$gmh \sin \theta+\varepsilon .\, \delta\theta=0$$

or

$$\delta\theta=-\frac{gmh}{\varepsilon}\sin \theta$$

The elasticity of the joint might be measured by holding the first piece of the pendulum firmly in the horizontal position, while the second piece stood out straight, and measuring the angular flexure at the joint. Denoting this by $\omega$, the last equation gives

$$\omega=\frac{gmh}{\varepsilon}$$

* The term *potential energy* grates upon the ear of a student of Aristotelian philosophy, because both words derive their whole standing in language from that philosophy, and in their proper meanings are directly contradictory of one another. "Energy" means actuality, and "potential" means not yet actualized, so that potential energy is unactual actuality. The conception of half the *vis viva* as an actuality or energy of which the negative of the potential is the corresponding potency or power, seems preferable to the contrary conception that the latter is the real "work" or performance, and the former merely the "power" or *vis* which develops it; because the negative of the potential may subsist for any length of time, but always tends to produce *vis viva*, while the *vis viva* must increase or diminish and does not particularly tend to do either rather than the other. I shall, therefore, venture to call $\frac{1}{2}\Sigma mv^2$ the kinetic act or kinetic energy and the negative of the potential, the kinetic power or kinetic potency. For the sum of the two I can think of no better term than *motivity* or *kinesis*.

Neglecting the square of $\theta$ (and *a fortiori* of $\omega$)

$$\delta\theta = -\omega\theta = -\omega\theta_0 \qquad \theta = (1-\omega)\theta_0$$

To find the kinetic energy, we assume a system of rectangular co-ordinates having the origin at the middle of the knife-edge, the axis of $y$ being directed vertically downwards, and that of $x$ sideways, perpendicular to the knife-edge. Then, writing the co-ordinates of the center of mass of the first piece $x_0$ and $y_0$, those of that of the second piece $x$ and $y$, we have

$$x_0 = h_0 \sin\theta_0 \qquad y_0 = h_0 \cos\theta_0$$

$$x = r\sin\theta_0 + h\sin\theta \qquad y = r\cos\theta_0 + h\cos\theta$$

$$(D_t x_0)^2 + (D_t y_0)^2 = h_0^2 (D_t \theta_0)^2 \qquad (D_t x)^2 + (D_t y)^2 = r^2 (D_t\theta_0)^2 + 2\, r\, h \cos(\theta-\theta_0) \,.\, D_t\theta_0 \,.\, D_t\theta + h^2 (D_t\theta)^2$$

The kinetic energy is, therefore,

$$E = [\tfrac{1}{2}\, m_0 (h_0^2 + \gamma_0^2) + \tfrac{1}{2}\, m\, r^2]\, (D_t\theta_0)^2 + mrh\cos\delta\theta \,.\, D_t\theta_0 \,.\, D_t\theta + \tfrac{1}{2}\, m\, (h^2+\gamma^2)\, (D_t\theta)^2$$

Neglecting the square of $\theta$, the differential equations are

$$[m_0 (h_0^2 + \gamma_0^2) + mr^2]\, D_t^2\theta_0 + (gm_0 h_0 + gm\, L + \varepsilon)\, \theta_0 + mrh\, D_t^2\, \theta - \varepsilon\theta = 0$$

$$mrh\, D_t^2\theta_0 - \varepsilon\theta_0 + m\, (h^2+\gamma^2)\, D_t^2\theta + (gmh+\varepsilon)\, \theta = 0.$$

Using the notation

$$A = m_0 (h_0^2 + \gamma_0^2) + mr^2 \qquad B = gm_0 h_0 + gm\, L + \varepsilon \qquad C = mrh \qquad E = -\varepsilon$$

$$G = m\, (h^2+\gamma^2) \qquad H = gmh + \varepsilon \qquad L = AG - C^2 \qquad M = AH + BG - 2CE \qquad N = BH - E$$

we have the two approximate values

$$D_t^2 = -\frac{N}{M} \text{ or } -\frac{M}{L}$$

If we measure the elasticity by holding the second piece in a rigidly horizontal position and observing the angular sagging at the joint due to the weight of the first piece, we have for the value of this angle

$$\psi = \frac{gmh}{\varepsilon}$$

Substituting we get

$$B = g\, (m_0 h_0 + mr + mh\, \psi^{-1}) \qquad E = -gmh\, \psi^{-1} \qquad H = gmh\, (1+\psi^{-1})$$

Then

$$M = gmh\, [m_0 (h_0^2 + \gamma_0^2) + mr^2]\, (1+\psi^{-1}) + gm\, (h^2+\gamma^2)\, (m_0 h_0 + mr + mh\, \psi^{-1}) + 2\, gm^2\, rh^2\, \psi^{-1}$$

$$\psi M = gmh\, [m_0 (h_0^2 + \gamma_0^2) + m\, (h+r)^2 + m\gamma^2] + gmh\, [m_0 (h_0^2+\gamma_0^2) + mr^2 + \frac{h^2+\gamma^2}{h} (m_0 h_0 + mr)]$$

$$N = g^2 mh\, (m_0 h_0 + mr + mh\psi^{-1})\, (1+\psi^{-1}) - g^2 m^2 h^2 \psi^{-2}$$

$$\psi N = g^2 mh\, [m_0 h_0 + m\, (h+r)] + g^2 mh\, (m_0 h_0 + mr)\, \psi$$

$$L = m\, (h^2+\gamma^2)\, [m_0 (h_0^2+\gamma_0^2) + mr^2] - m^2 r^2 h^2 = m\, (h^2+\gamma^2)\, m_0 (h_0^2+\gamma_0^2) + m^2 \gamma^2 r^2$$

If we now write

$$M = m_0 + m$$
$$MH = m_0 h_0 + m\, (h+r)$$
$$MHL = m_0 (h_0^2 + \gamma_0^2) + m\, [(r+h)^2 + \gamma^2]$$

and if we denote the period of oscillation by $T+\Delta T$, then when $\psi$ vanishes

$$T=\pi\sqrt{\frac{L}{g}}$$

and

$$(T+\Delta T)^2=\frac{\pi^2}{g}\,\frac{MHL+\left\{MHL+\left(m_0\frac{h_0}{h}-m\right)(h^2+\gamma^2)+mr\left(\frac{\gamma^2}{h}-h\right)\right\}\psi}{MH+[MH-mh]\,\psi}$$

$$\frac{\Delta T}{T}=\frac{1}{2}\left\{\frac{\left(m_0\frac{h_0}{h}-m\right)(h^2+\gamma^2)+mr\left(\frac{\gamma^2}{h}-h\right)}{MHL}+\frac{mh}{MH}\right\}\psi$$

If there are a number of stiff joints, the sum of their separate effects gives their combined effect.

It appears, therefore, that the effect of the flexure of a pendulum upon its period of oscillation is virtually to lengthen it by a quantity which is generally of the same order of magnitude as the amount by which one extremity of the pendulum would sag if the other end were held rigidly horizontal. This must be quite a considerable quantity for all the reversible pendulums which have ever been constructed.