APPENDIX No. 15.

ON THE USE OF THE NODDY FOR MEASURING THE AMPLITUDE OF SWAYING IN A PENDULUM SUPPORT.

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The "Noddy" is an instrument invented by Thomas Hardy, a well-known English clockmaker of the early part of this century. It was employed by him and others to detect any oscillatory swaying of a pendulum support, but I use it to measure the amplitude of such swaying. It consists of a little pendulum supported, like an ordinary clock pendulum, from a reed or spring; but instead of hanging down it stands erect, so that gravity acts against the spring and causes the pendulum, although quite short, to oscillate with the same natural period as the gravity pendulum, which hangs from and sways the support. The instrument that was constructed after my design by the late M. Breguet is firmly attached to a brass bed-plate, resting on three screw-feet. The reed is 1 centimeter in length, and from it springs the staff of the noddy, consisting of a steel wire 1 millimeter in diameter and about 9 centimeters long. From the lower part of the staff, just above the attachment of the reed, two short wires extend at right angles to the axis of the staff and in the plane of the oscillation; they have screw-threads cut upon them, and carry short brass cylinders through the axes of which they pass, and which make the principal weight of the noddy. They can be screwed along the wires so as to adjust the equilibrium and alter the radius of gyration about an axis through the center of mass. Another weight, spherical in form, slides upon the staff of the noddy, and serves to adjust the height of the center of mass. There are several sets of cylindrical weights and several sliding weights. At the top of the staff is fixed a small oblong frame carrying a glass scale of tenths of millimeters, the lines being vertical and the scale running in the direction of the oscillatory motion. The scale is 10 centimeters from the attachment of the staff to the reed. A pillar attached to the bed-plate, with its axis in the vertical plane of the reed, carries a horizontal microscope directed toward the scale on the noddy, which is illuminated from an adjustable reflector behind. The microscope is focused with a ratchet; it is furnished with a draw-tube, and carries in the focus of the eye-piece a horizontal scale or glass, each division of which is equivalent to about $0^{mm}.03$ in the focus of the objective as ordinarily used. The noddy is protected from currents of air by being inclosed in a tight metallic cylinder, furnished with two plate-glass windows opposite to one another at the level of the micrometer scale. This cylinder carries the reflector. It is also furnished with a stop-cock, so that the air can be exhausted if desired. The upper ends of the screws of the screw-feet are pointed like the lower ends, so as to serve as feet; and there is a wooden stand with rests for these feet, upon which the whole apparatus can be placed upside down to permit the observation of the period of oscillation of the pendent noddy.

We may first consider the case of the free oscillation of the noddy, and for the present we may neglect all resistance to its motion. Let us assume a system of rectangular co-ordinates having its origin at the root or fixed attachment of the reed, the axis of y being directed vertically upwards and that of x being in the plane of oscillation. The axis of z is then parallel to the axis of rotation, but the motion will be assumed to be in the plane of xy. Let s be the distance of any particle of the reed from the root; θ, the inclination of the reed to the vertical at the distance s from the root; S, the whole length of the reed; θ_s, the value of θ when $s=S$; θ_0, the value of θ when $s=0$, so that $\theta_0=0$; h, the distance of the center of mass of the noddy from the attachment of the reed; x and y, the co-ordinates of this center of mass; γ, the radius of gyration of the noddy about an axis parallel to that of z and passing through the center of mass; ε, the elasticity of the reed at any point, and we suppose this to be constant throughout the length of the reed; g, the acceleration of gravity; M, the mass of the noddy (that of the reed being neglected); E, the kinetic energy; Γ, the kinetic potency or positional energy. Then

$$x=\int_0^S \sin\theta \,.\, ds+h\sin\theta_s \qquad y=\int_0^S \cos\theta \,.\, ds+h\cos\theta_s$$

$$D_t x=\int_0^S \cos\theta \,.\, D_t\, ds+h\cos\theta_s \,.\, D_t\theta_s$$

$$D_t y=-\int_0^S \sin\theta \,.\, D_t \,.\, ds-h\sin\theta_s \,.\, D_t\theta_s$$

Let θ' be a quantity which is the same function of a variable s' that θ is of s. Then we have

$$\frac{E}{M}=\tfrac{1}{2}\int^S \left\{ ds'\int_0^S \cos(\theta-\theta') \,.\, D_t\theta \,.\, D_t\theta' \,.\, ds \right\} + h\, D_t\theta_s \,.\, \int_0^S \cos(\theta-\theta_s) \,.\, D_t\theta \,.\, ds+\tfrac{1}{2}(h^2+\gamma^2)(D_t\theta_s)^2$$

$$\frac{\Gamma}{M}=\frac{E}{M}\int_0 (D_s\theta)^2\, ds+g\int_0^S \cos\theta \,.\, ds+gh\cos\theta_s$$

In the first approximation we neglect the fourth power of θ in comparison with the second, and with this simplification we proceed to form the Lagrangian equations, according to the formula

$$D_t\frac{\delta E}{\delta D_t\theta}+\frac{\delta\Gamma}{\delta\theta}=0$$

The partial differential coefficients are to be taken on the hypothesis of a change in the value of θ corresponding to a single value of s, all other values remaining unchanged, so that

$$\frac{\delta\int F\theta \,.\, ds}{\delta D_t\theta}=ds \,.\, \frac{\delta F\theta}{\delta D_t\theta}$$

The partial differential coefficient of the first term of $\frac{E}{M}$ is

$$\tfrac{1}{2}\, ds'\int_0^S D_t\theta \,.\, ds+\tfrac{1}{2}\, ds\int_0^S D_t\theta' \,.\, ds'=ds\int_0^S D_t\theta \,.\, ds$$

This does not, however, apply to $\theta=\theta_s$; in that case the whole effect is given by the second and third terms of $\frac{E}{M}$. The partial differential coefficient of the first term of $\frac{\Gamma}{M}$ is most clearly deduced as follows: Let s_{i-1}, s_i, s_{i+1} be the distances of successive particles of the reed from the root, and let θ_{i-1}, θ_i, θ_{i+1} be the corresponding values of θ. We have

$$s_{i+1}-s_i=s_i-s_{i-1}=ds$$

Let us write

$$\frac{\varepsilon}{ds}=\eta$$

Then that part of the first term of $\frac{\Gamma}{M}$ which involves θ_i is

$$\tfrac{1}{2}\frac{\eta}{M}(\theta_{i+1}-\theta_i)^2+\tfrac{1}{2}\frac{\eta}{M}(\theta_i-\theta_{i-1})^2$$

and the differential coefficient of this, relatively to θ_i is

$$\frac{\eta}{M}(-\theta_{i+1}+2\,\theta_i-\theta_{i-1})=-\frac{\varepsilon}{M}D_s^2\,\theta\,.\,ds$$

But when $s=S$, the first term of the binomial expressing the part of $\frac{\Gamma}{M}$ to be considered is to be struck off, because there is no particle of the reed further from the root; consequently the differential coefficient then becomes

$$\frac{\eta}{M}(\theta_s-\theta_{s-1})=\frac{\varepsilon}{M}D_s\theta$$

We now see that the simplified Lagrangians are

$$D_t^2\int_0^s\theta\,.\,ds+h\,D_t^2\,\theta_s-\frac{\varepsilon}{M}D_s^2\,\theta-g\,\theta=0$$

$$h\,D_t^2\int_0^s\theta\,.\,ds+(h^2+\gamma^2)\,D_t^2\,\theta_s+\frac{\varepsilon}{M}D_s\,\theta_s-gh\,\theta_s=0$$

Differentiating the first of these equations relatively to s, we have

$$-\frac{\varepsilon}{M}D_s^3\,\theta-g\,D_s\,\theta=0$$

If we write σ for $\sqrt{\frac{\varepsilon}{Mg}}$, which has the dimension of a line, the solution to the last equation is

$$\theta=\Theta_1\sin\frac{s}{\sigma}+\Theta_2\cos\frac{s}{\sigma}+\Theta_3$$

where Θ_1, Θ_2, Θ_3, are arbitrary functions of the time, independent of s. Since $\theta_0=0$, $0=\Theta_2+\Theta_3$; so that

$$\theta=\Theta_1\sin\frac{s}{\sigma}+\Theta_2\left(\cos\frac{s}{\sigma}-1\right)$$

It thus appears that the figure of the reed is a curve of sines, or a part of such a curve, the wave-length being $\frac{\sigma}{2\pi}$.

We now form from the last equation expressions for θ_s and for $\int_0^s\theta\,.\,ds$, as well as for those terms of the two Lagrangians which involve θ and its derivatives; and from this we eliminate Θ_1 and Θ_2, so as to make the Lagrangians linear equations in θ_s and $\int_0^s\theta\,.\,ds$. And here it will be convenient to introduce the abbreviations

$$\chi=\frac{\int_0^s\theta\,.\,ds}{\sigma}\qquad \vartheta=\theta_s\qquad \psi=\frac{S}{\sigma}\qquad p=\sigma\cos\frac{S}{\sigma}-h\sin\frac{S}{\sigma}\qquad q=h\cos\frac{S}{\sigma}+\sigma\sin\frac{S}{\sigma}$$

The expression just found for θ then gives us

$$-\vartheta+\Theta_1\sin\psi-\Theta_2\,(1-\cos\psi)=0\qquad -\chi+\Theta_1\,(1-\cos\psi)-\Theta_2\,(\psi-\sin\psi)=0$$

$$\frac{\varepsilon}{M}D_s^2\,\theta+g\,\theta=-\Theta_2\,g\qquad \frac{\varepsilon}{Mh}D_s\,\theta_s=\Theta_1\,\frac{g}{h}\,\sigma\cos\frac{S}{\sigma}-\Theta_2\,\frac{g}{h}\,\sigma\sin\frac{S}{\sigma}$$

$$-g\,\theta_s=-\Theta_1\,\frac{g}{h}\,h\sin\frac{S}{\sigma}+\Theta_2\,\frac{g}{h}\,(h-h\cos\frac{S}{\sigma})$$

Eliminating Θ_1 and Θ_2 first from the first three of these equations, and afterward from the first two and the sum of the last two, we have

$$\begin{vmatrix} \frac{\varepsilon}{M} D_s^2 \theta + g\,\theta, & 0 & , g \\ \vartheta & , \sin\psi & , (1-\cos\psi) \\ \chi & , (1-\cos\psi), & \psi - \sin\psi \end{vmatrix} = 0$$

$$\begin{vmatrix} -\frac{\varepsilon}{Mh} D_s \theta_s + g\,\theta_s, -\frac{g}{h} p & , g - \frac{g}{h} q \\ \vartheta & , \sin\psi & , (1-\cos\psi) \\ \chi & , (1-\cos\psi), & \psi - \sin\psi \end{vmatrix} = 0$$

We have only to replace the first element of each of these determinants by its value as given by one of the Lagrangians, namely

$$\frac{\varepsilon}{M} D_s^2\theta + g\theta = hD_t^2\vartheta + \sigma D_t^2\chi \qquad -\frac{\varepsilon}{Mh} D_s\theta_s + g\theta_s = \left(h \mp \frac{\gamma^2}{h}\right) D_t^2\vartheta + \sigma D_t^2\chi$$

to obtain the Lagrangians freed from indeterminate values of s and θ. The two Lagrangians may now be embraced in a single expression by the introduction of an indeterminate number, n. Namely, we multiply the first by $(1-nh)$ and the second by nh, and add, when we get

$$\begin{vmatrix} (h+n\gamma^2) D_t^2\vartheta + \sigma D_t^2\chi, & -\ ngp, & g - ngq \\ \vartheta & , \sin\psi, & 1-\cos\psi \\ \chi & , 1-\cos\psi, & \psi - \sin\psi \end{vmatrix} = 0$$

The abscissa, x_r, of a particle on the staff of the noddy, at a distance r above the center of mass is

$$x_r = \sigma\chi + (h+r)\vartheta$$

Let ρ be the value of r for a particle so situated that it has a single harmonic motion. Then x^ρ being the abscissa,

$$D_t^2 x\rho = -\frac{\pi^2}{T^2} x\rho$$

where T is the period of oscillation. We may give n such a value that the equation combining the Lagrangians becomes identically equal to this, that is, to

$$(h+\rho) D_t^2\vartheta + \rho D_t^2\chi + \frac{\delta^2}{T^2}(h+\rho)\vartheta + \frac{\delta^2}{T^2}\sigma\chi = 0$$

This gives

$$\rho = n\gamma^2$$

$$\frac{\pi^2}{T^2}(h+\rho) = g \frac{\begin{vmatrix} 1-\cos\psi, & \psi - \sin\psi \\ -\ \ np, & 1-nq \end{vmatrix}}{\begin{vmatrix} \sin\psi, & 1-\cos\psi \\ 1-\cos\psi, & \psi - \sin\psi \end{vmatrix}} = g \frac{1-\cos\psi + n(h-q+p\psi)}{\psi\sin\psi - 2(1-\cos\psi)}$$

$$\frac{\pi^2}{T^2}\sigma = g \frac{\begin{vmatrix} -np, & 1-nq \\ \sin\psi, & 1-\cos\psi \end{vmatrix}}{\begin{vmatrix} \sin\psi, & 1-\cos\psi \\ 1-\cos\psi, & \psi - \sin\psi \end{vmatrix}} = g \frac{n(\sigma-p) - \sin\psi}{\psi\sin\psi - 2(1-\cos\psi)}$$

I have carefully performed the elimination of ρ and n from these equations, and have thus obtained the quadratic

$$\frac{\pi^4}{g^2T^4}\gamma^2\sigma\left[2(1-\cos\psi) - \psi\sin\psi\right] - \frac{\pi^2}{gT^2} - (h^2+\gamma^2+\sigma^2)\sin\psi - \sigma p\psi + p = 0$$

The proper adjustment of the noddy requires p to be a very short length. The coefficient of the first term ought also to be small. Then the solution of the quadratic is

$$\frac{\delta^2}{g\mathrm{T}^2}=\frac{1}{2}\frac{(h^2+\gamma^2+\sigma^2)\sin\psi-\sigma p\psi}{\gamma^2\sigma\,(2\text{ ver sin }\psi-\psi\sin\psi)}\left(1\pm\sqrt{1-4\frac{\gamma^2\sigma p\,(2\text{ ver sin }\psi-\psi\sin\psi)}{\{(h^2+\gamma^2+\sigma^2)\sin\psi-\sigma p\psi\}^2}}\right)$$

and the approximate values of the roots are

$$\frac{\delta^2}{g\mathrm{T}_1^2}=\frac{(h^2+\gamma^2+\sigma^2)\sin\psi-\sigma p\psi}{\gamma^2\sigma\,(2\text{ ver sin }\psi-\psi\sin\psi)}=\frac{h^2+\gamma^2+\sigma^2+\mathrm{S}h-\mathrm{S}\sigma\cot\psi}{\gamma^2\,(2\sigma\tan\frac{1}{2}\psi-\mathrm{S})}$$

$$\frac{\delta^2}{g\mathrm{T}_2^2}=\frac{p}{(h^2+\gamma^2+\sigma^2)\sin\psi-\sigma p\psi}=\frac{\frac{\sigma}{h}\cot\psi-1}{\frac{h^2+\gamma^2+\sigma^2}{h}+\mathrm{S}-\mathrm{S}\frac{\sigma}{h}\cot\psi}$$

The latter root represents the principal component of the oscillation. The corresponding values of ρ are

$$\rho_1=-\frac{(h^2+\sigma^2)\sin\psi-\sigma p\psi}{\sigma-p}=-\frac{h+\frac{\sigma^2}{h}-\mathrm{S}\left(\frac{\sigma}{h}\cot\psi-1\right)}{1+\frac{\sigma}{h}\tan\frac{1}{2}\psi}\qquad\rho_2=\gamma^2\frac{(h^2+\gamma^2+\sigma^2)\sin\psi-2\sigma p\tan\frac{1}{2}\psi}{(h^2+\gamma^2+\sigma^2+\mathrm{S}h-\mathrm{S}\sigma\cot\psi)\,(\sigma-p)}$$

For any fixed value of ψ, the first component oscillation will be infinitely rapid when

$$2\sigma\tan\tfrac{1}{2}\psi-\mathrm{S}=0$$

that is, when

$$\tan\frac{\mathrm{S}}{2\sigma}=\frac{\mathrm{S}}{2\sigma}$$

and the second component oscillation will have a period of infinite length when

$$\frac{\sigma}{h}\cot\psi-1=0$$

that is, when

$$\tan\frac{\mathrm{S}}{\sigma}=\frac{\sigma}{h}$$

This affords a means of determining σ, by measuring h when the adjustment is such as to give this condition of things.

The amplitudes of the two component oscillations depend upon the manner in which the noddy is set into motion, but the second will usually be the principal one and the first will be insensible; the noddy will consequently rotate about a fixed point determined by the value of ρ_1.

When the noddy is in the pendent position the vertical co-ordinates may be taken to increase downwards. Then, those terms of $\frac{\Upsilon}{\mathrm{M}}$ which involve g will have their signs reversed. The equation to determine the figure of the reed will accordingly be

$$\frac{\varepsilon}{\mathrm{M}}\mathrm{D}_s^3\,\theta-g\,\mathrm{D}_s\,\theta=0$$

The solution of this contains gudermannian instead of trigonometric functions, and may be written

$$\theta=-\Theta_1\sinh\frac{s}{\sigma}+\Theta_2\cosh\frac{s}{\sigma}+\Theta_3$$

and since as before

$$\theta_0=0=\Theta_2+\Theta_3$$

this takes the form

$$\theta = -\Theta_1 \sinh \frac{s}{\sigma} + \Theta_2 \left(\cosh \frac{s}{\sigma} - 1\right)$$

This gives us

$$\vartheta + \Theta_1 \sinh \psi - \Theta_2 \left(\cosh \psi - 1\right) = 0$$

$$\chi + \Theta_1 \left(\cosh \psi - 1\right) - \Theta_2 \left(\sinh \psi - \psi\right) = 0$$

$$+\frac{\varepsilon}{M} D_s^2 \theta - g\,\theta - g\,\Theta_2 = 0$$

$$-\frac{\varepsilon}{Mh} D_s \theta_s - g\,\theta_s - \Theta_1 \frac{g}{h} p^1 + \Theta_2 \left(+\frac{g}{h} q^1 - g\right) = 0$$

where

$$p^1 = h \sinh \psi + \sigma \cosh \psi$$
$$q^1 = h \cosh \psi + \sigma \sinh \psi$$

Thus the Lagrangians become

$$\begin{vmatrix} h\, D_t^2 \vartheta + \sigma D_t^2 \chi, & 0 & , +g \\ \vartheta & , \sinh \psi & , \cosh \psi - 1 \\ \chi & , \cosh \psi - 1, & \sinh \psi - \psi \end{vmatrix} = 0$$

$$\begin{vmatrix} \left(h + \frac{\gamma^2}{h}\right) D_t^2 \vartheta + \sigma D_t^2 \chi, & -\frac{g}{h} p^1, & -\frac{g}{h} q^1 + g \\ \vartheta & , \sinh \psi, & \cosh \psi - 1 \\ \chi & , \cosh \psi - 1, & \sinh \psi - \psi \end{vmatrix} = 0$$

and their combination is

$$\begin{vmatrix} (h + n\gamma^2) D_t^2 \vartheta + \sigma D_t^2 \chi, & -ngp' & , g - ngq' \\ \vartheta & , \sinh \psi & , \cosh \psi - 1 \\ \chi & , \cosh \psi - 1, & \sinh \psi - \psi \end{vmatrix} = 0$$

The equations to determine ρ and T are

$$\rho = n\gamma^2,$$

$$\frac{\pi^2}{T^2}(h + n\gamma^2) = g \frac{\begin{vmatrix} \cosh \psi - 1, & \sinh \psi - \psi \\ -np' & , 1 - nq' \end{vmatrix}}{\begin{vmatrix} \sinh \psi & , \cosh \psi - 1 \\ \cosh \psi - 1, & \sinh \psi - \psi \end{vmatrix}} = g \frac{\cosh \psi - 1 + n\,(-h + q' - p'\psi)}{2\,(\cosh \psi - 1) - \psi \sinh \psi}$$

$$\frac{\pi^2}{T^2}\sigma = g \frac{\begin{vmatrix} -np' & , 1 - nq' \\ \sinh \psi & , \cosh \psi - 1 \end{vmatrix}}{\begin{vmatrix} \sinh \psi & , \cosh \psi - 1 \\ \cosh \psi - 1, & \sinh \psi - \psi \end{vmatrix}} = g \frac{-\sinh \psi + n\,(p' - \sigma)}{2\,(\cosh \psi - 1) - \psi \sinh \psi}$$

The elimination of n gives

$$\frac{\pi^4}{g^2 T^4}\gamma^2 \sigma \{2\,(\cosh \psi - 1) - \psi \sinh \psi\} - \frac{\pi^2}{gT^2}\{(h^2 + \gamma^2 - \sigma^2) \sinh \psi - \sigma p' \psi + 2h\sigma\,(\cosh \psi - 1)\} + p' = 0$$

and the approximate values of the roots are

$$\frac{\pi^2}{gT_1^2} = \frac{(h^2 + \gamma^2 - \sigma^2) \sinh \psi - \sigma p' \psi + 2h\sigma\,(\cosh \psi - 1)}{\gamma^2 \sigma\,[2\,(\cosh \psi - 1) - 4 \sinh \psi]}$$

$$\frac{\pi^2}{gT_2^2} = \frac{p'}{(h^2 + \gamma^2 - \sigma^2) \sinh \psi - \sigma p' \psi + 2h\sigma\,(\cosh \psi - 1)} = \frac{1 + \frac{\sigma}{h} \coth \psi}{\frac{h^2 + \gamma^2 - \sigma^2}{h} - S - s\frac{\sigma}{h} \coth \psi + 2\sigma \tanh \frac{1}{2}\psi}$$

When the noddy stands on the support of a gravity pendulum oscillating in the same plane, we may neglect the influence of the former upon the latter. Then if ξ be the horizontal displacement of the support, we have

$$D_t x = \int_0^s \cos\theta \,.\, D\,\theta \,.\, ds + h\cos\vartheta \,.\, D\,\vartheta + D_t\xi$$

Consequently $\frac{E}{M}$ is increased by the terms

$$D_t\xi \int_0^s \cos\,\theta . D_t\,\theta \,.\, ds + h\cos\vartheta \,.\, D_t\vartheta \,.\, D\,\xi + \tfrac{1}{2}\,(D_t\,\xi)^2$$

The first Lagrangian heretofore considered will be increased by $D_t^2\,\xi$, and the second by $h\,D_t^2\,\xi$. The figure of the reed will not be affected, and the combination of the Lagrangians will simply have $D_t^2\,\xi$ added to it. We will now write

$$\xi = \Xi\cos\frac{t}{T'}\pi$$

where Ξ is the constant amplitude of oscillation of the support and T′ is the period of the gravity pendulum. Thus, the differential equation for x_ρ becomes

$$D_t^2\,[(h+\rho)\,\vartheta+\sigma\,\chi]+\frac{\pi^2}{T^2}[(h+\rho)\vartheta+\sigma\,\chi]-\frac{\pi^2}{T'^2}\,\Xi\cos\frac{t}{T'}\pi=0$$

This will add to the motion of x_ρ a harmonic component, having the period T′, so that it will be

$$(h+\rho)\,\vartheta+\sigma\,\chi=X\cos\frac{t-t_0}{T}\pi-Q\cos\frac{t}{T'}\pi$$

To determine Q we take the second derivative:

$$D_t^2\,[(h+\rho)\,\vartheta+\sigma\,\chi]=-\frac{\pi^2}{T^2}X\cos\frac{t-t_0}{T}\pi+\frac{\pi^2}{T'^2}Q\cos\frac{t}{T'}\pi$$

$$=-\frac{\pi^2}{T^2}[(h+\varsigma)\,\vartheta+\sigma\,\chi+\frac{\pi^2}{T'^2}\Xi\cos\frac{t}{T'}$$

$$=-\frac{\pi^2}{T^2}X\cos\frac{t-t_0}{T}\pi+\pi^2\left(\frac{Q}{T^2}+\frac{\Xi}{T'^2}\right)\cos\frac{t}{T'}\pi$$

Thus we have

$$\frac{Q}{T'^2}=\frac{Q}{T^2}+\frac{\Xi}{T'^2}$$

or

$$Q=\frac{T^2}{T^2-T'^2}\,\Xi$$

But the noddy has no oscillation to begin with. This fact is represented by the equations

$$t_0=0 \qquad\qquad X=Q$$

Hence

$$(h+\rho)\,\vartheta+\sigma\chi=\frac{T^2\,\Xi}{T^2-T'^2}\left(\cos\frac{t}{T}\pi-\cos\frac{t}{T'}\pi\right)=\Xi\,\frac{2T^2}{T^2-T'^2}\sin\frac{T-T'}{2TT'}t\pi\,\sin\frac{T+T'}{2TT'}t\pi$$

This equation shows that the noddy would oscillate with a period, a sort of mean between its natural period and that of the gravity pendulum. The amplitude of oscillation would increase from nothing at an initial rate not much affected by the value of (T − T′) until it would reach its maximum, when

$$t=\frac{TT'}{T-T'}$$

At the beginning the noddy would be a quarter of a phase behind the gravity pendulum; at the maximum oscillation of the noddy it would be in opposition to the pendulum; and when it was reduced to rest again it would be a quadrant in advance. It would then start up as before.

In considering the influence of the gravity pendulum upon the noddy, however, it is essential to take account of the resistance to the motion of the latter, owing to the internal friction of the spring and to the viscosity of the air. The dissipation produced by the former cause will be

$$\tfrac{1}{2}\,\mu\int_0^S (D_t\,\theta)^2\,ds$$

where μ is a constant. This will add the term

$$\frac{\mu}{M}D_t\,\theta$$

to the first Lagrangian. It will slightly change the figure of the spring, and the equation to determine this will be a partial differential equation, showing that the wave-length will not be constant. But this effect will be very small and may be neglected. Neglecting also the effect of the resistance upon the period of the motion, we find that if the natural motion of the noddy is

$$(h+\rho)\,\vartheta+\sigma\chi=\Theta\,e^{\frac{t}{2T'}B\pi}\cos\frac{t}{T}\pi$$

then its motion under the influence of the pendulum is

$$(h+\rho)\,\vartheta+\sigma\chi=\frac{\Xi}{R}\left\{\sin\omega\,.\,e^{-\frac{t}{2T'}B\pi}\cos\frac{t}{T}\pi+\sin\left(\frac{t}{T'}\pi-\omega\right)\right\}$$

where

$$\tan\omega=\frac{A}{B}\qquad R=\sqrt{A^2+B^2}\qquad A=1-\frac{T'^2}{T^2}$$

It will be seen that the natural period and rate of decrement of the arc of the noddy have to be observed, and that weighings and measures of its parts have to be made so as to calculate $\rho_1-\rho_2$. Then, it is necessary to observe, while the gravity pendulum is swinging, the relative amplitude and phase of the motion of the noddy.

I have made considerable use of the instrument, and find it gives results agreeing within a few *per cent.*, and that it is on the whole a tolerably satisfactory way of determining the amount of swaying of a pendulum support.