APPENDIX No. 14.

NOTE ON THE THEORY OF THE ECONOMY OF RESEARCH, BY ASSISTANT C. S. PEIRCE.

When a research is of a quantitative nature, the progress of it is marked by the diminution of the probable error. The results of non-quantitative researches also have an inexactitude or indeterminacy which is analogous to the probable error of quantitative determinations. To this inexactitude, although it be not numerically expressed, the term "probable error" may be conveniently extended.

The doctrine of economy, in general, treats of the relations between utility and cost. That branch of it which relates to research considers the relations between the utility and the cost of diminishing the probable error of our knowledge. Its main problem is, how, with a given expenditure of money, time, and energy, to obtain the most valuable addition to our knowledge.

Let r denote the probable error of any result, and write $s = \frac{1}{r}$. Let $\mathrm{U}r \,.\, dr$ denote the infinitesimal utility of any infinitesimal diminution, dr, of r. Let $\mathrm{V}s \,.\, ds$ denote the infinitesimal cost of any infinitesimal increase, ds, of s. The letters U and V are here used as functional symbols. Let subscript letters be attached to r, s, U, and V, to distinguish the different problems into which investigations are made. Then, the total cost of any series of researches will be

$$\Sigma_i \int \mathrm{V}_i s_i \,.\, ds_i\,;$$

and their total utility will be

$$\Sigma_i \int \mathrm{U}_i r_i \,.\, dr_i\,.$$

The problem will be to make the second expression a maximum by varying the inferior limits of its integrations, on the condition that the first expression remains of constant value.

The functions U and V will be different for different researches. Let us consider their general and usual properties. And, first, as to the relation between the exactitude of knowledge and its utility. The utility of knowledge consists in its capability of being combined with other knowledge so as to enable us to calculate how we should act. If the knowledge is uncertain, we are obliged to do more than is really necessary, in order to cover this uncertainty. And, thus, the utility of any increase of knowledge is measured by the amount of wasted effort it saves us, multiplied by the specific cost of that species of effort. Now, we know, from the theory of errors, that the uncertainty in the calculated amount of effort necessary to be put forth may be represented by an expression of the form

$$c\sqrt{a + r^2}\,,$$

where a and c are constants. And, therefore, the differential coefficient of this, multiplied by the specific cost of the effort in question, say $\frac{h}{c}$, gives

$$\mathrm{U}r = h\frac{r}{\sqrt{a + r^2}}\,.$$

When a is very small compared with r this becomes nearly constant, and in the reverse case it is nearly proportional to r. An analogous proposition must hold for non-quantitative research.

Let us next consider the relation between the exactitude of a result and the cost of attaining it. When we increase our exactitude by multiplying observations, the different observations being independent of one another as to their cost, we know from the theory of errors that $\int \mathrm{V}s \,.\, ds$ is proportional to s^2, and that consequently $\mathrm{V}s$ is proportional to s. If the costs of the different observations are not independent (which usually happens), the cost will not increase so fast relatively to the accuracy; but if the errors of the observations are not independent (which also usually happens), the cost will increase faster relatively to the accuracy; and these two perturbing influences may be supposed, in the long run, to balance one another. We may, therefore, take $Vs = ks$, where k represents the specific cost of the investigation.

We thus see that when an investigation is commenced, after the initial expenses are once paid, at little cost we improve our knowledge, and improvement then is especially valuable; but as the investigation goes on, additions to our knowledge cost more and more, and, at the same time, are of less and less worth. Thus, when chemistry sprang into being, Dr. Wollaston, with a few test tubes and phials on a tea-tray, was able to make new discoveries of the greatest moment. In our day, a thousand chemists, with the most elaborate appliances, are not able to reach results which are comparable in interest with those early ones. All the sciences exhibit the same phenomenon, and so does the course of life. At first we learn very easily, and the interest of experience is very great; but it becomes harder and harder, and less and less worth while, until we are glad to sleep in death.

Let us now apply the expressions obtained for Ur and Vs to the economic problem of research. The question is, having certain means at our disposal, to which of two studies they should be applied. The general answer is that we should study that problem for which the economic urgency, or the ratio of the utility to the cost

$$\frac{Ur \,.\, dr}{Vs \,.\, ds} = r^2 \frac{Ur}{Vs} = \frac{h}{k} \; \frac{r^4}{\sqrt{a + r^2}}$$

is a maximum. When the investigation has been carried to a certain point this fraction will be reduced to the same value which it has for another research, and the two must then be carried on together, until finally, we shall be carrying on, at once, researches into a great number of questions, with such relative energies as to keep the urgency-fraction of equal values for all of them. When new and promising problems arise they should receive our attention to the exclusion of the old ones, until their urgency becomes no greater than that of others. It will be remarked that our ignorance of a question is a consideration which has between three and four times the economic importance of either the specific value of the solution or the specific cost of the investigation in deciding upon its urgency.

In order to solve an economical problem, we may use as variables

$$x = \int Vs \,.\, ds,$$

or the total cost of an inquiry, and

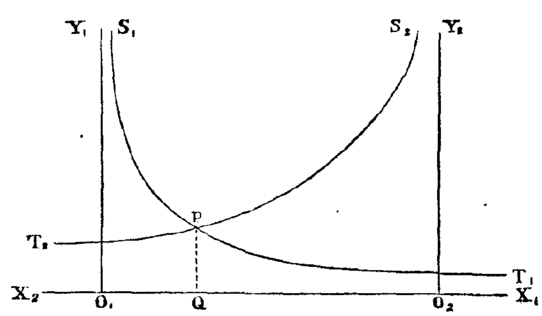
$$y = \frac{Ur \,.\, dr}{Vs \,.\, ds},$$

or the economic urgency. Then, C being the total amount we have to spend in certain researches, our equations will be

$$C = x_1 + x_2 + x_3 + \text{etc.}$$
$$y_1 = y_2 = y_3 = \text{etc.}$$

Then, expressing each y in terms of x, we shall have as many equations as unknown quantities.

When we have to choose between two researches only, the solution may be represented graphically, as follows:



From any point O_1 taken as an origin, draw the axis of abscissas $O_1 X_1$, along which x_1, the total cost of the first investigation, is to be measured. Draw also the axis of ordinates $O_1 Y_1$, along which y_1, the economic urgency of the first investigation, is to be measured. Draw the curve $S_1 T_1$ to represent the relations of x_1 and y_1. Take, on the axis $O_1 X_1$, a point O_2 such that $O_1 O_2$ shall measure the total cost of the two investigations. Let x_2, the total cost of the second investigation, be measured on the same axis as x_1, but in the opposite direction. From O_2 draw the axis of ordinates $O_2 Y_2$ parallel to $O_1 Y_1$, and measure y_2, the economic urgency of the second investigation, along this axis. Draw the curve $S_2 T_2$ to represent the relations of x_2 and y_2. Then, the two curves $S_1 T_1$ and $S_2 T_2$ will generally cut one another at one point, and only one, between the axes $O_1 Y_1$ and $O_2 Y_2$. From this point, say P, draw the ordinate P Q, and the abscissas $O_1 Q$ and $O_2 Q$ will measure the amounts which ought to be expended on the two inquiries.

According to the usual values of U and V, we shall have

$$y = \frac{1}{4} \, \frac{h\,k}{x\sqrt{a\,x^2 + \frac{1}{2}k\,x}}\,.$$

In this case, when there are two inquiries, the equation to determine x_1 will be a biquadratic. Two of its roots will be imaginary, one will give a negative value of either x_1 or x_2, and the fourth, which is the significant one, will give positive values of both.

Let us now consider the economic relations of different researches to one another. 1st, as alternative methods of reaching the same result, and 2d, as contributing different premises to the same argument.

Suppose we have two different methods of determining the same quantity. Each of these methods is supposed to have an accidental probable error and a constant probable error, so that the probable errors, as derived from n observations in the two ways, are:

$$r_1 = \sqrt{R_1^2 + \frac{\rho_1^2}{n}} \quad \text{and} \quad r_2 = \sqrt{R_2^2 + \frac{\rho_2^2}{n}}\,.$$

The probable error of their weighted mean is

$$\frac{1}{\sqrt{\frac{1}{r_1^2} + \frac{1}{r_2^2}}}\,,$$

if their constant probable errors are known. The sole utility of any observation of either is to reduce the error of the weighted mean; hence,

$$U\,r_1 = D_{r_1}(r_1^{-2} + r_2^{-2})^{-\frac{1}{2}} = (r_1^{-2} + r_2^{-2})^{-\frac{3}{2}} r_1^{-3}.$$

And as the cost is proportional to the number of observations

$$V\,s_1 = k_1 \frac{1}{D_{n_1} s_1} = \frac{k_1}{D_{n_1}(R_1^2 + \rho_1^2 n_1^{-1})^{-\frac{1}{2}}} = \frac{2\,k_1\,r_1^3\,n_1^2}{\rho_1^2}\,.$$

Hence, the urgency is (omitting a factor common to the values for the two methods)

$$r_1^2 \frac{U\,r_1}{V\,s_1} = \frac{}{k_1 \rho_1^2 \left(1 + n_1 \frac{R_1^2}{\rho_1^2}\right)^2}\,.$$

And, as the urgency of the two methods ought to be the same at the conclusion of the work, we should have

$$\sqrt{k_1}\,.\,\rho_1\left(1 + n_1 \frac{R_1^2}{\rho_1^2}\right) = \sqrt{k_2}\,.\,\rho_2\left(1 + n_2 \frac{R_2^2}{\rho_2^2}\right),$$

which equation serves to determine the relative values of n_1 and n. We again perceive that the cost is the smallest consideration. The method which has the smallest accidental probable error is the one which is to be oftenest used in case only a small number of observations are made; but if a large number are taken the method with the larger accidental probable error is to be oftenest used, unless it has so much greater a probable constant error as to countervail this consideration. If one of the two methods has only $\frac{1}{p}$th the accidental probable error of the other, but costs p^2 times as much, the rule should be to make the total cost of the two methods inversely proportional to the squares of their constant errors.

Let us now consider the case in which two quantities x_1 and x_2 are observed, the knowledge of which serves only to determine a certain function of them, y. In this case the probable error of y is

$$\sqrt{\mathrm{D}_{x_1} y \cdot r_1^2 + \mathrm{D}_{x_2} y \cdot r_2^2}\,,$$

and we shall have

$$\mathrm{U} r_1 = 2 r_1 \frac{\mathrm{d}y}{\mathrm{d}x_1}$$

$\mathrm{V}s_1$ will have the same value as before; but neglecting now the constant error, we may write

$$\mathrm{V} s_1 = 2 k_1\, \rho_1\, n_1^{\frac{1}{2}}.$$

Then the urgency (with omission of the common factor) is

$$\frac{\rho_1^2}{k_1\, n_1^2} \cdot \frac{\mathrm{d}y}{\mathrm{d}x_1},$$

and, as the two urgencies must be equal, we have

$$\frac{n_1}{n_2} = \frac{\rho_1}{\rho_2} \sqrt{\frac{k_2}{k_1}} \sqrt{\frac{\frac{\mathrm{d}y}{\mathrm{d}x_1}}{\frac{\mathrm{d}y}{\mathrm{d}x_2}}}\,.$$

The following is an example of the practical application of the theory of economy in research: Given a certain amount of time, which is to be expended in swinging a reversible pendulum, how much should be devoted to experiments with the heavy end up, and how much to those with the heavy end down?

Let T_d be the period of oscillation with heavy end down, T_u the same with heavy end up. Let h_d and h_u be the distances of the center of mass from the points of support of the pendulum in the two positions. Then the object of the experiments is to ascertain a quantity proportional to

$$h_d\, \mathrm{T}_d - h_u\, \mathrm{T}_u\,.$$

Accordingly, if dT_d and dT_u are the probable errors of T_d and T_u, that of the quantity sought will be

$$\sqrt{h_d^2\, (\mathrm{d\,T}_d)^2 + h_u^2\, (\mathrm{d\,T}_u^2)}\,.$$

We will suppose that it has been ascertained, by experiment, that the whole duration of the swinging being C, and the excess of the duration of the swinging with heavy end down over that with heavy end up being x, the probable errors of the results are

$$\mathrm{dT}_d = \sqrt{a + \left(b + \frac{c}{h_d^2}\right)\frac{1}{\mathrm{C}+x}}$$

$$\mathrm{dT}_u = \sqrt{a + \left(b + \frac{c}{h_u^2}\right)\frac{1}{\mathrm{C}-x}}$$

where a, b, and c are constants. Then, the square of the probable error of the quantity sought will be

$$a\,(h_d^2 + h_u^2) + (b\,h_d^2 + c)\frac{1}{\mathrm{C}+x} + (b\,h_u^2 + c)\,\frac{1}{\mathrm{C}-x}.$$

The differential coefficient of this relatively to x is

$$-(b\,h_d^2 + c)\frac{1}{(\mathrm{C}+x)^2} + (b\,h_u^2 + c)\,\frac{1}{(\mathrm{C}-x)^2}\,.$$

Putting this equal to zero and solving, we find for the only significant root,

$$\frac{x}{C} = \frac{b\,(h_d^2 + h_u^2) + 2c}{b\,(h_d^2 - h_u^2)} - \sqrt{\left(\frac{b\,(h_d^2 + h_u^2) + 2c}{b\,(h_d^2 - h_u^2)}\right)^2 - 1}$$

when b vanishes, x reduces to zero, and the pendulum should be swung equally long in the two positions. When c vanishes, as it would if the pendulum experiment were made absolutely free from certain disturbing influences, we have

$$\frac{x}{C} = \frac{h_d - h_u}{h_d + h_u},$$

so that the duration of an experiment ought to be proportional to the distance of the center of mass from the point of support. This would be effected by beginning and ending the experiments in the two positions with the same amplitudes of oscillation.

It is to be remarked that the theory here given rests on the supposition that the object of the investigation is the ascertainment of truth. When an investigation is made for the purpose of attaining personal distinction, the economics of the problem are entirely different. But that seems to be well enough understood by those engaged in that sort of investigation.