Plan of a New Reversible Pendulum

The exterior form of the pendulum will be that of a cylinder 4 cm in diameter and 1.5 metre long, terminated at the two ends by hemispheres. There will be four small projections upon the cylinder upon two of which it rests at a time in oscillating.

The pendulum is to carry no knife-edges. In place of that it is to rest on knife-edges. The parts resting on the knife-edges are not to be ~~cylind~~ planes but cylinders having radii of 5 centimetres. These cylindrical pieces are to be of agate. They are to be one centimetre square. They are to be bevelled and set into pieces of brass 2cm × 1cm provided with screw holes by which they can be screwed onto bracket-shaped pieces on the pendulum.

The distance from agate to agate is to be precisely one metre at 15°C.

$$\delta^2 + \frac{2\sqrt{a}(a-b)+3ai+j}{5a-b+3\sqrt{a}.i}\delta + \frac{a\sqrt{a}.i+\sqrt{a}.j}{5a-b+3\sqrt{a}.i} = 0$$

$$\frac{20a^2\sqrt{a}.i - 4ab\sqrt{a}.i + 20a\sqrt{a}.j - 4b\sqrt{a}.j + 12a^2i^2 + 12aij}{4(5a-b+3\sqrt{a}.i)^2}$$

$$4a^3+8a^2b+4ab^2 + 12a^2\sqrt{a}.i - 12ab\sqrt{a}.i + 4\sqrt{a}.(a-b)j$$

$$+9a^2i^2+6aij+j^2$$

$$\left(\delta + \frac{2\sqrt{a}(a-b)+3ai+j}{10a-2b+6\sqrt{a}.i}\right)^2 = \frac{4a^3+8a^2b+4ab^2-8a^2\sqrt{a}.i-8ab\sqrt{a}.i-16a\sqrt{a}j+3a^2i^2-6aij+j^2}{4(5a-b+3\sqrt{a}.i)^2}$$

$$\delta = \frac{-2\sqrt{a}(a-b)-3ai-j}{10a-2b+6\sqrt{a}.i} \pm \frac{\sqrt{4a^3+8a^2b+4ab^2-8a^2\sqrt{a}.i-8ab\sqrt{a}.i-16a\sqrt{a}.j-3a^2i^2-6aij+j^2}}{10a-2b+6\sqrt{a}.i}$$

The shorter pendulum will usually oscillate at nearly its ~~own~~ nearest ~~tim~~ period of oscillation on the same

$$5a-b\left(1+\frac{3\sqrt{a}}{5a-b}i\right)\delta^2$$

$$+2\sqrt{a}(a-b)\left(1+\frac{3a+\frac{j}{i}}{2\sqrt{a}(a-b)}i\right)\delta$$

$$+a\sqrt{a}.i+\sqrt{a}.j=0$$

$$\begin{array}{l}-6a(a-b)\\+15a^2-3ab+(5a-b)\frac{j}{i}\\\hline 9a^2+3ab+(5a-b)\frac{j}{i}\\\hline 2\sqrt{a}(a-b)(5a-b)\\\cancel{+5a^2\sqrt{a}-ab\sqrt{a}+(5a-b)\sqrt{a}\frac{j}{i}}\end{array}$$

$$\delta^2+2\frac{\sqrt{a}(a-b)}{5a-b}\left(1+\frac{9a^2+3ab+5a-b)\frac{j}{i}}{2\sqrt{a}(a-b)(5a-b)}i\right)$$

$$+\frac{a\sqrt{a}+\sqrt{a}\frac{j}{i}}{5a-b}i\left(1-\frac{3a.i}{5a-b}\right)=0$$

$$\frac{1}{2}(I_2+I_1)\varepsilon\left(1+\frac{(I_2-I_1)(I_2\gamma_1+I_1\gamma_2)}{(I_2+I_1)^2\,\varepsilon}+\frac{1}{2}\left(\frac{I_2\gamma_1+I_1\gamma_2}{(I_2+I_1)\,\varepsilon}\right)^2-\frac{1}{2}\frac{(I_2-I_1)^2(I_2\gamma_1+I_1\gamma_2)^2}{(I_2+I_1)^4\,\varepsilon^2}\right)$$

$$=\frac{1}{2}(I_2+I_1)\varepsilon+\frac{1}{2}\frac{I_2-I_1}{I_2+I_1}(I_2\gamma_1+I_1\gamma_2)+\frac{I_1I_2}{(I_2+I_1)^2}(I_2\gamma_1+I_1\gamma_2)^2\frac{1}{(I_2+I_1)\varepsilon}$$

Consequently the differential operator becomes

$$I_1I_2D_t^2+\{$$

$$\left\{(I_2+I_1)\varepsilon+\frac{I_2^2\gamma_1-I_1^2\gamma_2}{I_2+I_1}+\frac{I_1I_2}{(I_2+I_1)^2}(I_2\gamma_1+I_1\gamma_2)^2\frac{1}{(I_2+I_1)\varepsilon}\right\}\times$$

$$\left\{I_1I_2D_t^2+\frac{I_1I_2}{I_1+I_2}(\gamma_1-\gamma_2)-\frac{I_1I_2}{(I_2+I_1)^2}(I_2\gamma_1+I_1\gamma_2)^2\frac{1}{(I_1+I_2)\varepsilon}\right\}$$

The first factor relates merely to a vibration in the joint. The second alone need be considered. It shows that the square of the period of oscillation is lengthened in the ratio

$$1+\frac{(I_2\gamma_1+I_1\gamma_2)^2}{(I_1+I_2)^2(\gamma_1-\gamma_2)\varepsilon}$$

The value of ε will be experimentally determined by holding the upper part of the pendulum in a vise in the horizontal position and observing the depression of the lower part produced by adding a known load at a known distance.

~~If we~~ In order to get ~~an idea~~ a general conception of the mode of motion of the pendulum, suppress the first factor of the differential equation and write

$$I_1 D_t^2 \varphi_1 + \left\{ \frac{I_1}{I_1+I_2}(\gamma_1-\gamma_2) - \frac{I_1}{(I_1+I_2)^3}\frac{(I_2\gamma_1+I_1\gamma_2)^2}{\varepsilon} \right\} \varphi_1 = 0$$

Subtract this from the first equation

$$I_1 D_t \varphi_1 = -\gamma_1\varphi_1 - \varepsilon(\varphi_1-\varphi_2)$$

and we have

$$\frac{I_1}{I_1+I_2}\left\{(\gamma_1-\gamma_2) - \frac{(I_2\gamma_1+I_1\gamma_2)^2}{(I_1+I_2)^2\varepsilon}\right\}\varphi_1 = (\varepsilon+\gamma_1)\varphi_1 - \varepsilon\varphi_2$$

or

$$\varphi_2-\varphi_1 = \frac{1}{\varepsilon}\left\{\frac{I_2\gamma_1+I_1\gamma_2}{(I_1+I_2)\varepsilon} + \frac{I_1}{(I_1+I_2)}\frac{(I_2\gamma_1+I_1\gamma_2)^2}{\varepsilon^2}\right\}\varphi_1$$

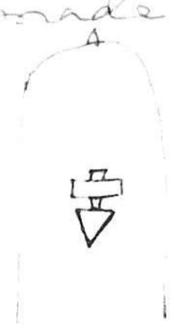
This shows that, except for a vibrating motion which may be set up, its upper point will always be a little more displaced than the lower in a fixed ratio.

$$1 + \left(\frac{I_1\gamma_2 - I_2\gamma_1}{I_1+I_2}\right)^2 \frac{1}{(\gamma_1+\gamma_2)^2} = \frac{\dfrac{\gamma_1\varphi_1+\gamma_2\varphi_2}{\gamma_1+\gamma_2}}{\dfrac{I_1\varphi_1+I_2\varphi_2}{I_1+I_2}}$$

We thus have ~~an~~ another expression for the effect of the bending, which shows that φ_1 and φ_2 are in a constant ratio. The last expression is also approximately equal to

$$1 + \frac{\gamma_1-\gamma_2}{\gamma_1+\gamma_2} - \frac{I_1-I_2}{I_1+I_2}\,\frac{\varphi_1-\varphi_2}{\varphi_1+\varphi_2}$$

A metre Repsold reversible pendulum was firmly held by the heavy bob in a horizontal position and 2 Kilo-grams was laid on the middle point. Then the knife-edge at the further end was found to be depressed by the weight by 0.7 mm. The total ~~weight~~ mass of the pendulum is 6.3 Kilos. The ~~weight~~ mass of the heavy bob is about one third of this, and we may say that the mass of the long part of the pendulum is 4 Kilos. Hence, when $\varphi_1+\varphi_2=3.14$ or π we have $\varphi_1-\varphi_2 = 0.0014$; so that in the absolute measure. Still, it may be taken as a matter of course, I think, that the new pendulums should not be open to the same objection. To avoid it, I propose to use only two centimetres of the knife, projecting from the tube at the two ends; and thus avoid cutting anything but a small pair of holes in the tube, at the same time strengthening it at the points where the holes are made. The sketch will give an idea.



A triangular knife to traverse the tube and be held in place by a small screw pressing against the back of it.

Complete solution is

$$\varphi_1 = \beta_1 A \cos \frac{t - \tau_1}{\sqrt{\frac{\lambda_1+\lambda_2}{2} + \sqrt{\left(\frac{\lambda_1-\lambda_2}{2}\right)^2 + \beta_1\beta_2}}} + \beta_1 B \cos \frac{t-\tau_2}{\sqrt{\frac{\lambda_1+\lambda_2}{2} - \sqrt{\left(\frac{\lambda_1-\lambda_2}{2}\right)^2+\beta_1\beta_2}}}$$

$$\varphi_2 = \left\{\sqrt{\left(\frac{\lambda_1-\lambda_2}{2}\right)^2+\beta_1\beta_2} - \frac{\lambda_1-\lambda_2}{2}\right\} A \cos \frac{t-\tau_1}{\sqrt{\frac{\lambda_1+\lambda_2}{2} + \sqrt{\left(\frac{\lambda_1-\lambda_2}{2}\right)^2+\beta_1\beta_2}}}$$

$$- \left\{\sqrt{\left(\frac{\lambda_1-\lambda_2}{2}\right)^2+\beta_1\beta_2} + \frac{\lambda_1-\lambda_2}{2}\right\} B \cos \frac{t-\tau_2}{\sqrt{\frac{\lambda_1+\lambda_2}{2} - \sqrt{\left(\frac{\lambda_1-\lambda_2}{2}\right)^2+\beta_1\beta_2}}}$$

Condition that long pendulum shall have short period of oscillation is that

$$B > A$$

Max. amplitude $\varphi_1 = \beta_1(A+B)$

Min amplitude φ_2

$$= (\sqrt{\ } + \delta) B - (\sqrt{\ } - \delta) A = \sqrt{\ }(B-A) + \delta(B+A)$$

Then minimum ampl. φ_2 must exceed $\frac{\delta}{\beta_1} \times$ max ampl. φ_1

They must at ~~some~~ their moment have opposite phases

3	2	1	0
4444	$\frac{1}{4}$ 444N	$\frac{1}{4}$ 44N4	$\frac{1}{6}$ 4N4N
NNNN	$\frac{1}{4}$ N444	$\frac{1}{4}$ 4N44	$\frac{1}{6}$ N4N4
	$\frac{1}{6}$ 44NN	$\frac{1}{6}$ 4NN4	
	$\frac{1}{6}$ NN44	$\frac{1}{6}$ N44N	
	$\frac{1}{4}$ 4NNN	$\frac{1}{4}$ N4NN	
	$\frac{1}{4}$ ~~4NNN~~ NNN4	$\frac{1}{4}$ NN4N	

2 $1\frac{1}{3}$

$\frac{6}{15}$ $\frac{4}{15}$

$$\sqrt{\frac{b_1 b_2}{g_1 g_2}}\sqrt{1 + \quad l+a_1}$$

$$g_1 = g + \gamma \qquad b_1 = b + \beta$$
$$g_2 = g - \gamma \qquad b_2 = b - \beta$$

$$\frac{b^2 - \beta^2}{g^2 - \gamma^2} \qquad \frac{b^2\left(1 - \frac{\beta^2}{b^2}\right)}{g^2} + \frac{\gamma^2}{g^2}\Big)$$

$$\frac{b}{g}\left(1 - \frac{1}{2}\frac{\beta^2}{b^2} + \frac{1}{2}\frac{\gamma^2}{g^2}\right)$$

$$g\, 1 - \frac{1}{16}\varphi_1^2$$
$$+ \quad \frac{1}{16} g\left(\varphi_1^2 - \varphi_2^2\right)$$

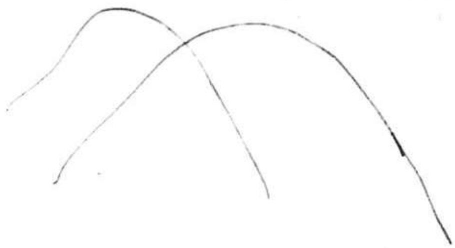
$$\frac{l+\lambda+a+\alpha}{2(g+\gamma)} + \frac{l-\lambda+a-\alpha}{2(g-\gamma)}$$

$$\frac{l+a+\lambda+\alpha}{2g}\left(1 - \frac{\gamma}{g} + \frac{\gamma^2}{g^2}\right) + \frac{l+a-\lambda-\alpha}{2g}\left(1 + \frac{\gamma}{g} + \frac{\gamma^2}{g^2}\right) =$$

$$\frac{l+a}{g} - \frac{(\lambda+\alpha)\gamma}{g^2} + \frac{(l+a)\gamma^2}{g^3} \qquad \varphi = \frac{1}{100}$$

$$\frac{1}{10{,}000}$$

$$\frac{l}{g} + \frac{a-b}{g} - \frac{(\lambda+\alpha)\gamma}{g^2} + \frac{1}{2}\frac{\beta^2}{g b} + \frac{l\gamma^2}{g^3} - \frac{(a - \frac{1}{2}b)\gamma^2}{g^3}$$



$$\partial_t^2 \varphi = -\frac{g}{\ell}\varphi$$

$$\varphi = A\cos\left(\sqrt{\frac{g}{\ell}}\,(t-t_0)\right)$$

$$\lambda_1\lambda_2 - \beta_1\beta_2 - \lambda_2^2$$

$$\left(\frac{\lambda_1-\lambda_2}{2}\lambda_2 - \beta_1\beta_2 \pm \lambda_2\sqrt{\quad}\right)\varphi_1 =$$

$$\left(-\frac{\lambda_1\lambda_2-\beta_1\beta_2}{\lambda_1+\frac{\beta_1\beta_2}{\lambda_1-\lambda_2}} + \lambda_2\right)\varphi_1 = \beta_1\varphi_2$$

$\ominus$

$$= -\lambda_1\lambda_2 + \beta_1\beta_2 + \lambda_1\lambda_2 + \beta_1\beta_2\frac{\lambda_2}{\lambda_1-\lambda_2}$$

$$1+\frac{\lambda_2}{\lambda_1-\lambda_2}$$

$$\frac{\beta_1\beta_2\frac{\lambda_1}{\lambda_1-\lambda_2}}{\lambda_1+\frac{\beta_1\beta_2}{\lambda_1-\lambda_2}} \qquad \frac{\cancel{\beta_1\beta_2}\lambda_1}{\lambda_1(\lambda_1-\lambda_2)+\beta_1\beta_2} = \frac{\varphi_2}{\varphi_1} = \frac{1}{\frac{\lambda_1-\lambda_2}{\beta_2}+\frac{\beta_1}{\lambda_1}}$$

$$\beta_2 D_t^2 \varphi_1 + (\lambda_2 D_t^2 + 1)\varphi_2 \qquad -\frac{\cancel{\lambda_1}\beta_2}{\lambda_1-\lambda_2}\varphi_1 = -\cancel{\ }\varphi_2.$$

$$(\lambda_1 \cancel{D_t^2} + \cancel{D_t^2})\varphi_1 + \beta_1 \cancel{D_t^2}\varphi_2 = 0 \qquad (\lambda_1 + D_t^{-2})\varphi_1 \cancel{+\beta} = -\beta_1 \varphi_2$$

$$x\varphi_1 + \bar{x}\varphi_2$$

$$(\lambda_1 + 1)$$

$$(x b_1' + \bar{x} b_2)\bar{x} g_2 = (\bar{x} b_2' + x b_1) x g_1$$

$$\frac{x}{\bar{x}} \cancel{x} b_1' g_2 + \cancel{\bar{x}}^2 b_2 g_2 = \frac{x}{\bar{x}} \cancel{x} b_2' g_1 + \frac{x^2}{\bar{x}^2} b_1 g_1$$

$$\cancel{\xi^2 + \frac{b_2' g_1 + b_1' g_2}{b_2' g_1}} \qquad \xi = b_1 g_1 \frac{x}{\bar{x}}$$

$$\xi^2 + \xi(b_2' g_1 - b_1' g_2) = b_1 b_2 g_1 g_2$$

$$\xi = \tfrac{1}{2} b_1' g_2 - \tfrac{1}{2} b_2' g_1 \pm \sqrt{\tfrac{1}{4}(b_2' g_1 - b_1' g_2)^2 + b_1 b_2 g_1 g_2}$$

$$\frac{1-x}{x}$$

$$\frac{x}{\bar{x}} = \frac{1}{2}\frac{b_1'}{b_1}\frac{g_2}{g_1} - \frac{1}{2}\frac{b_2'}{b_1}$$

$$\frac{1}{x} - 1$$

$$1 - \frac{3}{2}(1 - \rho)$$

$$\frac{3}{2}\rho - \frac{1}{2}$$

$$\frac{3}{2}\rho M < \frac{1}{2} M$$

$$\rho < \frac{1}{3} \qquad \frac{3}{2} \text{[illegible]}$$

$$\rho = \frac{1}{3} + \lambda$$

$$\frac{3}{2}\rho = \frac{1}{2} + \frac{3}{2}\lambda \qquad \frac{3}{2}\lambda \text{ [illegible]}$$

$$\text{or} \quad \text{[illegible]} + \frac{3}{4}\lambda \qquad \lambda \text{ [illegible]}$$

$$\text{[illegible]} + \frac{3}{4}\left(\rho - \frac{1}{3}\right) = \frac{1}{4} + \frac{3}{4}\rho$$

$$= \frac{3}{2}\rho - \frac{3}{4}\left(\rho - \frac{1}{3}\right)$$