March 17, 1880.

(1) PROFESSOR SYLVESTER, On the Triangles in- and ex-scribable to a General Cubic Curve.

The general cubic being thrown into the form $xy^2 + yz^2 + zx^2 + mxyz = 0$, the lines $x = 0$, $y = 0$, $z = 0$ will constitute an in- and ex-scribable triangle to the curve. The number of such was stated to be 24; consisting of 12 pairs of conjugates, each pair being in triple-perspective position with respect to each other, and the centres of the perspective projection being three collinear points of inflexion. Accordingly, the twenty-four triangles will consist of 4 groups of three pairs of conjugate in- and ex-scripts. The law for the number of polygons in- and ex-scripts, with any assigned number of sides will be found stated in No. 4, Vol. II, of the *American Journal of Mathematics*.

(2) MR. MITCHELL: On Binomial Congruences (Second paper).

For the prime modulus, a, $x^n \equiv R$ (mod. a) has $\alpha + 1$ roots, where α is the g. c. d. of n and $\phi(a)$. R means a totient-residue and has here two values, 0 and 1. $X^n \equiv$ (mod. a^k) has $(\alpha+1)(\beta+1)$ roots, and so on for moduli, abc, &c. $X^n \equiv R$ (mod. a^l) has $\alpha + a^{l-\Gamma\binom{l}{n}}$ roots. $\Gamma\binom{l}{n}$ denotes the number of integers in $\frac{l}{n}$, plus one, if there be a fraction. R has two values, 0 and 1; for $R = 1$ the congruence has α roots; for $R = 0$, $a^{l-\Gamma\binom{l}{n}}$ roots.

Then $x^n \equiv R$ (mod. $a^l b^k c^i$) has $\left(\alpha + a^{l-\Gamma\binom{l}{n}}\right)\left(\beta + b^{k-\Gamma\binom{k}{n}}\right)\left(\gamma + c^{i-\Gamma\binom{i}{n}}\right)$ roots, and the different terms of the expansion give the number of roots for the 2^m separate values of R, m being the number of different prime factors in the modulus.

If one of the prime factors of the modulus, as a, $= 2$, l being greater than unity, 2, the α of the formula must be multiplied by 2 when it is greater than unity.

The first part of the theorem in respect to prime moduli given, for instance in Serret's *Cours D'Algèbre Supérieure*, Art. 310, can be generalized as follows:

Denoting by $\chi(M)$ the period of the modulus M, if $x^\theta \equiv R$, (mod. M), then $\xi^{\frac{\chi(M)}{\theta}} \equiv R$ (mod. M), θ being any divisor of $\chi(M)$. The reciprocal is not true in general. $x^{\chi(M)} - R \equiv 0$ can be written $(x^{\frac{\chi(M)}{2}} - R)(x^{\frac{\chi(M)}{2}} + R) \equiv 0$, since $R \equiv R^2$ (mod. M). The quadratic residues of M are all roots of $x^{\frac{\chi(M)}{2}} - R \equiv 0$, therefore all the roots of $x^{\frac{\chi(M)}{2}} + R \equiv 0$, are *non-quadratic* residues. Some of the *non-residues* may be roots of $x^{\frac{\chi(M)}{2}} - R \equiv 0$, and some may be roots of neither of the two latter congruences, but of their product.

Other theorems for prime moduli can be generalized for any modulus.

[For proofs, &c., See Journal of Math. Vol. III. No. 1].

(3) MR. STRINGHAM: On Rotation in Four-dimensional Space.

The rotation of a particle, or point, about a plane in four-dimensional space may be defined as such a motion of the particle that a radius vector drawn from a fixed point in the plane to the particle shall always remain constant in length and perpendicular to the plane.

If this radius vector be drawn perpendicular to the three-dimensional flat space in which the given plane is situated, since it is perpendicular to every line in that space, the plane may be rotated in that space, about the foot of the radius vector and still remain perpendicular to it; or, since the perpendicularity of the line to the plane depends on their *relative* positions alone, the line may also be rotated about the fixed point in the plane (the plane being fixed) in some *other* particular space of 3 dimensions than the one before considered and still remain perpendicular to the plane; that is to say

THEOREM I: *A rotation about a plane in four-dimensional space is equivalent to a rotation about a point in three-dimensional space.*

From this theorem it follows as a direct consequence that "if a fourth dimension were added to space, a closed material surface could be turned inside out without either stretching or tearing." [See Professor Newcomb's Note in Amer. Jour. of Math., Vol. I., No. 1, p. 1.] For a material point on one side of the surface will reverse its aspect on being rotated to the *opposite* side.

THEOREM II: *The four-dimensional surface generated by the revolution of a plane polygon about a fixed plane in four-dimensional space is equivalent to the three dimensional area of the projection of the polygon on the rotation plane, multiplied by the circumference of the circle whose radius is the length of the normal to the given polygon at its centre of gravity intercepted by the rotation plane.*

Let A = area of the given polygon, A′ = its projection on the rotation plane, n = the normal above described, p = the perpendicular let fall from the centre of gravity of the given polygon to the plane; then $\frac{n}{p} = \frac{A}{A'}$.

Thus if Z = area of four-fold zone generated by a complete revolution,

$$\therefore Z = 2\pi p. A = 2\pi n A'. \qquad \text{Q. E. D.}$$

Theorems similar to these may be proved for the higher-fold spaces.

If the 3-fold hemisphere of radius r be considered as made up of an infinite number of plane polygons, its projection on a diametral plane has for its area the area of a great circle, and theorem II. gives

$$S_{(4)} = 2\pi r.\ \pi r^2 = 2\pi^2 r^3,$$

as the area of the 4-fold sphere. [See Dr. Story's paper on n-dimensional spheres, in this *Circular*.] Comparing this with the formula

$$V_{(4)} = 2\pi^2 r^3 . \frac{r}{4} = \frac{1}{2}\pi^2 r^4$$

and carrying the process to the higher-fold spaces, it is easy to see that in general $S_{(n)} = 2\pi r\, V_{(n-2)}$.

The following formulae are thence easily deduced:

$$S_{(n)} = \frac{2\pi r^2}{(n-2)} S_{(n-2)},\quad V_{(n)} = \frac{2\pi r^2}{n} V_{(n-2)};\quad S_{(2n)} = \frac{2\pi^n r^{2n-1}}{(n-1)!},$$

$$S_{(2n+1)} = \frac{2\pi^n r^{2n}}{(n-\frac{1}{2})!};\quad V_{(2n)} = \frac{\pi^n r^{2n}}{n!},\quad V_{(2n+1)} = \frac{\pi^n r^{2n+1}}{(n+\frac{1}{2})!};$$

where $S_{(k)}$ means the surface and $V_{(k)}$ the volume of the k-fold sphere.

March 9, 1880.

(1) MR. PEIRCE: On Kant's "Critic of the Pure Reason" in the light of Modern Logic.

Mr. Peirce compared Kant's solution of the problem "How are synthetical judgments *à priori* possible?" with the solution given by modern logic of the problem "How are synthetical judgments in general possible?" He showed that the reply which Kant makes to the former question has its analogue with reference to the latter. This analogous answer to the second question is true, indeed, but is far from being a complete solution of the problem. On the other hand, the solution which modern logic gives of its question may be successfully applied to Kant's problem; but this does not enable us to discover the origin of the conceptions of space and time. The categories of Kant were next considered. The list given by him is built upon the basis of a formal logic which subsequent criticism has undermined and carried away. Nevertheless, there really do exist relationships between some of those conceptions and logic on the one hand and time on the other. The explanation of these relationships in conformity with modern logic, though far more definite than that of Kant, is not altogether dissimilar to it.

(2) MR. STRINGHAM gave a *resumé* of Dr. Ernst Schröder's *Operationskreis des Logikkalkuls*.

The main features of Schröder's system are, (1) its dualistic arrangement, by which to every addition equation a multiplication equation is made to correspond (and *vice versa*), (2) the transformation of every logical equation into a form in which the right hand member is zero, and (3) more especially his announcement of a *Hauptheorem* by means of which by a single operation two equations are obtained, from one of which any given class symbol is eliminated, while the other gives its value. The method of (2) is identical with that of Jevons.

The fundamental departure from Boole's system is in the definition of logical addition, wherein Schröder corrects the error of Boole, who assumed that logical addition presupposes the mutual exclusion of the class symbols added. This correction Schröder adopts from Robert Grassmann.

The following equations are important in Schröder's method and display its dual character. No. 20 with its five adjuncts is the *Hauptheorem*.

$$5^\circ \quad aa = a; \qquad | \quad 5' \quad a + a = a.$$
$$6^\circ \quad a(b + c) = ab + ac; \quad | \quad 6' \quad a + bc = (a + b)(a + c).$$
$$7^\circ \quad aa_1 = 0; \qquad | \quad 7' \quad a + a_1 = 1;$$

where a_1 means not a. Every class symbol may be put into either of the two forms

$$14^\circ \quad b = xa + ya_1; \ | \ 14' \quad b = (y + a)(x + a_1);$$

where x and y are arbitrary.

$$16^\circ \quad \text{If } a + b = 0, \ \therefore a = b = 0; \quad | \quad 16' \quad \text{If } ab = 1, \ \therefore a = b = 1.$$

The equation a = b may always be transformed into either of the forms, 17°, 17′:

$$17^\circ \quad ab_1 + a_1 b = 0; \ | \ 17' \quad ab + a_1 b_1 = 1.$$

The negations of ab and of (a + b) are

$$18^\circ \quad (ab)_1 = a_1 + b_1; \ | \ 18' \quad (a + b)_1 = a_1 b_1.$$

By means of 17° every logical equation involving the class symbol a may be put into the form

$$20 \quad xa + ya_1 = 0:$$

and from this equation are deduced any and all of the five relations,

$$xy = 0; \quad a = ux_1 + y;$$
$$a = (u + y)x_1, \ a = (uy_1 + y)x \quad a = ux_1 y_1 + y;$$

where u is an arbitrary class symbol.

An illustration of the application of this method was given by the solution of the question given by Boole—*Laws of Thought*, p. 146.

The preface to Dr. Schröder's book is dated March, 1877. Following this, the author gives a short bibliography of the subject in which he mentions the names of Trendelenburg, Boole, Cayley, Ellis and the two Grassmanns, Robert and Hermann. From this it is to be inferred that he was unacquainted with the investigations of the two writers who, some ten years before his own book was published, had made the most important additions to Boole's system, viz.: Peirce and Jevons. Hence, Schröder's work is chiefly a reproduction of what had already been done by these two writers. His *Hauptheorem*, however, is unique.

MR. PEIRCE remarked that Professor Jevons and himself, independently of each other, had corrected Boole's error concerning the mutual exclusion of classes in logical addition, the former in a pamphlet entitled "Pure Logic or the Logic of Quality," 1864, the latter in a paper on "An Improvement in Boole's Calculus of Logic," read before the *American Academy*, May, 1867; and that in the latter paper the dual character of the subject was plainly exhibited, a fact acknowledged by Schröder in a note in a recent number of *Königsberger's Repertorium*. Mr. Peirce then gave an illustration of his own method as applied to the solution of the question that had just been solved by Schröder's method.