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Robin Wilson: Four Colors Suffice

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2 *The problem is posed*

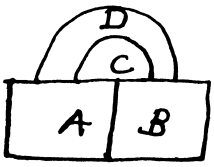
Unlike many problems in mathematics, the origin of the four-colour problem can be traced precisely – to a letter written in London in 1852. However, for many years it was believed that the problem could be traced back even further – to a lecture given in Germany around 1840. We start our historical narrative by investigating these rival claims and explaining how the confusion arose.

DE MORGAN WRITES A LETTER

On 23 October 1852, Augustus De Morgan, professor of mathematics at University College, London, wrote to his friend Sir William Rowan Hamilton, the distinguished Irish mathematician and physicist. This was nothing unusual. The two men had corresponded for many years, exchanging family news, reporting on the latest scientific gossip in London and Dublin, and sharing bits of mathematical news. Certainly, neither of them could have imagined that the contents of this particular letter would create mathematical history, for it was here that the *four-colour problem* was born.

A student of mine asked me to day to give him a reason for a fact which I did not know was a fact — and do not yet. He says that if a figure be any how divided and the compartments differently coloured so that figures with any portion of common boundary line are differently coloured — four colours may be wanted but not more — The following is his case in which four are wanted

A B C & D are names of Colours



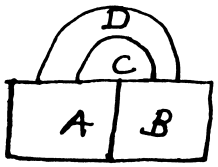
Query cannot a necessity for five or more be invented

Part of Augustus De Morgan's letter to Sir William Rowan Hamilton, 23 October 1852.

My dear Hamilton, . . .

A student of mine asked me to day to give him a reason for a fact which I did not know was a fact – and do not yet. He says that if a figure be any how divided and the compartments differently coloured so that figures with any portion of common boundary *line* are differently coloured – four colours may be wanted, but not more – the following is his case in which four *are* wanted

A B C D are names of colours



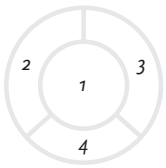
Query cannot a necessity for five or more be invented . . .

What do you say? And has it, if true been noticed? My pupil says he guessed it in colouring a map of England . . . The more I think of it the more evident it seems. If you retort with some very simple case which makes me out a stupid animal, I think I must do as the Sphynx did . . .

Doing as the Sphynx did would have been rather drastic. The Sphynx of ancient mythology was a legendary figure who leapt to her death after Oedipus had correctly solved a difficult riddle she had set him. The riddle was this: What animal walks on four legs in the morning, two at noon, and three in the evening? The answer is Man (as a baby, as an adult, and as an elderly person with a stick).

Years later, the student who had approached De Morgan that fateful day identified himself as Frederick Guthrie, subsequently a physics professor and founder of the Physical Society in London. But it was not Frederick who had coloured the map of England, as he recalled in 1880:

Some thirty years ago, when I was attending Professor De Morgan's class, my brother, Francis Guthrie, who had recently ceased to attend them (and who is now professor of mathematics at the South African University, Cape Town), showed me the fact that the greatest necessary number of colours to be used in colouring a map so as to avoid identity of colour in lineally contiguous districts is four. I should not be justified, after this lapse of time, in trying to give his proof, but the critical diagram was as in the margin.



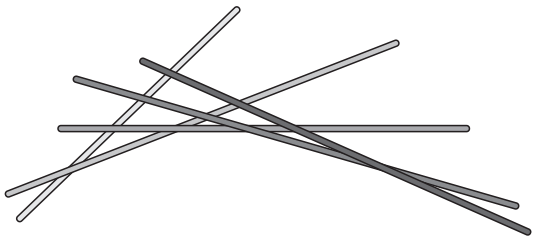
With my brother's permission I submitted the theorem to Professor De Morgan, who expressed himself very pleased with it; accepted it as new; and, as I am informed by those who subsequently attended his classes, was in the habit of acknowledging whence he had got his information.

If I remember rightly, the proof which my brother gave did not seem altogether satisfactory to himself; but I must refer to him those interested in the subject . . .

Thus it was Frederick Guthrie's elder brother Francis who could justly claim to have originated the four-colour problem, but the nature of the 'proof' he gave is not known. Francis Guthrie had been a former student of De Morgan's at University College, obtaining a Bachelor of Arts degree there in 1850. Two years later he took a Bachelor of Laws degree, and was called to the bar in 1857. He had a distinguished career in South Africa, becoming professor of mathematics at the newly established college at

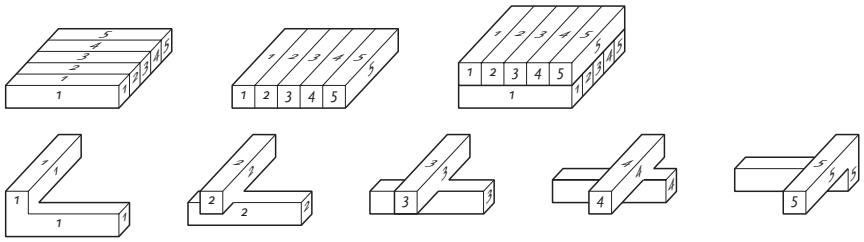
Graaff-Reinet in the Cape Colony, and later at the South African College in Cape Town. A well-liked and popular figure, Guthrie also contributed to botany, which became his chief hobby, and the plant *Guthriea capensis* and the heather *Erica guthriei* were named after him. But he never published anything on the colouring of maps or on the problem that is still sometimes referred to as *Guthrie's problem*.

Frederick Guthrie seems to have been the first to observe that the four-colour problem has no interesting extension to three dimensions: if we allow three-dimensional 'countries', then we can construct maps that require as many colours as we wish. An example, included by him in the note about his brother, involves a collection of flexible rods (or pieces of coloured wool), all touching each other. Since each rod must have a different colour from all those it touches, we need as many colours as there are rods: for example, the five rods in Guthrie's diagram below require five colours.



Another three-dimensional example, later described by the Austrian mathematician Heinrich Tietze, involves taking a number of horizontal bars numbered 1 to *n*, placing on top of them *n* vertical bars also numbered 1 to *n*, and then joining (as a single 'country') each pair of horizontal and vertical bars with the same number. We then obtain *n* three-dimensional countries, all touching each other, which therefore require *n* colours. Here, *n* can be as large as

we wish: the following pictures show how to construct the five countries when $n = 5$.



HOTSPUR AND THE *ATHENAEUM*

By 1852 Augustus De Morgan and Sir William Rowan Hamilton were both well established in their respective careers. De Morgan had studied at Cambridge University before becoming the first professor of mathematics at the newly founded University College in London, a position he held for over thirty years. An eccentric and prolific writer with a style all his own, he is mainly remembered for his popular book *A Budget of Paradoxes*, for 'De Morgan's laws' in set theory and for his contributions to mathematical logic. Hamilton was a child prodigy, familiar with Latin, Greek and Hebrew at the age of five, and speaking Arabic, Sanskrit, Turkish and other languages by the time he was fourteen. He became Astronomer Royal of Ireland while still an undergraduate at Trinity College, Dublin, and held this position until his death in 1865.

As we remarked earlier, De Morgan's 1852 letter to Hamilton was not an isolated event, for they corresponded regularly for thirty years. They met only once, around 1830, when they were introduced to each other by Charles Babbage, whose designs for the so-called Analytical Engine foreshadowed the invention of the programmable computer a century later. There was a second

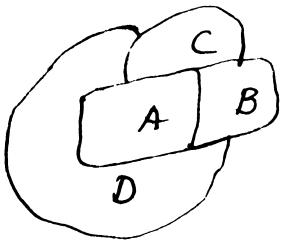
Augustus De Morgan (1806–71)

occasion on which De Morgan and Hamilton were both present, a Freemasons' dinner in honour of the astronomer and mathematician Sir John Herschel, but the event was so crowded that they had no chance to speak to each other.

When De Morgan wrote to Hamilton about the map-colour problem, he doubtless hoped that Hamilton would become interested in it. After all, De Morgan had taken an interest in Hamilton's researches, including Sir William's ground-breaking work on *quaternions*. Many mathematical operations are *commutative*, which is to say that they can be carried out in either order: for example, the addition and multiplication of ordinary numbers are commutative ($3 + 4 = 4 + 3$ and $3 \times 4 = 4 \times 3$). However, for Hamilton's quaternions, multiplication is not commutative: his 'numbers' are the sum of four terms $a + bi + cj + dk$ (where a, b, c, d are numbers and $i^2 = j^2 = k^2 = -1$) that multiply in a non-commutative way: for example, $i \times j = k$ but $j \times i = -k$, and $k \times i = j$ but $i \times k = -j$.

In the event, De Morgan's letter to Hamilton about the map-colour problem drew a terse and idiosyncratic reply: 'I am not likely to attempt your ''quaternion'' of colours very soon.' Undeterred, De Morgan wrote to other mathematical friends trying to interest them in the problem. He was fascinated by its intricacies, and in his original letter to Hamilton he had tried to explain where the difficulty lies:

> As far as I see at this moment, if four *ultimate* compartments have each boundary line in common with one of the others, three of them inclose the fourth, and prevent any fifth from connexion with it. If this be true, four colours will colour any possible map without any necessity for colour meeting colour except at a point.
>
> Now, it does seem that drawing three compartments with common boundary *A B C* two and two – you cannot make a fourth take boundary from all, except inclosing one – But it is tricky work

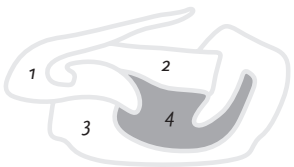


and I am not sure of all convolutions – What do you say? And has it, if true been noticed?

In this passage De Morgan hits upon the fact that if a map contains four regions, each adjoining the other three, then one of them must be completely enclosed by the others. He believed, incorrectly, that this idea lay at the heart of the problem, and it soon became an obsession of his. Since he could not prove it, he proposed to assume its truth as an *axiom*, which he defined as 'a proposition which cannot be made dependent upon obviously more simple ones'.

In December 1853, De Morgan wrote to the distinguished philosopher William Whewell, Master of Trinity College, Cambridge, describing his observation as a mathematical axiom that had lain 'wholly dormant' until it arose in connection with the map-colour problem:

I soon made out the following – which was at first incredible – then certainly true – then axiomatic – for I cannot make it depend on anything I see more clearly.



If *four* non-interfering compartments have *each* common boundary *line* with the other three – one at least of the four must be inclosed by the other three – or by fewer . . .

Six months later, in a letter to the Cambridge mathematician Robert Ellis, De Morgan further described it as

an instance of Whewell's views about *latent* axioms, things which at first are not even credible, but which settle down into first principles.

The first known appearance of the four-colour problem in print was also connected with William Whewell. On 14 April 1860, a lengthy unsigned review of Whewell's book *The Philosophy of Discovery, Chapters Historical and Critical* appeared in the *Athenaeum*, a popular literary journal of the time. In his review, the writer outlined the four-colour problem, claiming that the problem was familiar to cartographers. He followed his description with a very obscure passage:

Now, it must have been always known to map-colourers that four different colours are enough. Let the counties come cranking in, as Hotspur says, with as many and as odd convolutions as the designer chooses to give them; let them go in and out and roundabout in such a manner that it would be quite absurd in the Queen's writ to tell the sheriff that A. B. could run up and down in his bailiwick: still, four colours will be enough to make all requisite distinction.

This mention of Hotspur refers to a passage in Shakespeare's *King Henry IV, Part 1*, where Hotspur remarks, 'See how this river comes me cranking in . . .'.

The reviewer then asserted that if four areas on a map all have a boundary with the other three, then one area must be surrounded by the others, and this passage clearly identifies De Morgan as the author of the review. De Morgan had indeed written to Whewell on

3 March 1860, thanking him for sending a copy of his book and informing him that he had received a further copy from the *Athenaeum* for review, 'which will go back uncut' (in those days one needed a paper cutter to separate the pages of a book).

As a consequence of De Morgan's *Athenaeum* review, the four-colour problem crossed the Atlantic to the USA. There it was perused by the American mathematician, philosopher and logician Charles Sanders Peirce (pronounced 'purse'), who developed a lifelong interest in the problem. Peirce considered it 'a reproach to logic and to mathematics that no proof had been found of a proposition so simple', and subsequently presented an attempted solution at Harvard University in the presence of his father Benjamin Peirce, the distinguished Harvard professor of mathematics and natural philosophy. As Charles Peirce later wrote:

> About 1860 De Morgan in the *Athenaeum*, called attention to the fact that this theorem had never been demonstrated; and I soon after offered to a mathematical society at Harvard University a proof of this proposition extending it to other surfaces for which the numbers of colors are greater. My proof was never printed, but Benjamin Peirce, J. E. Oliver, and Chauncey Wright, who were present, discovered no fallacy in it.

In fact, the seminar probably took place in the late 1860s, but the Peirce manuscripts at Harvard University do not indicate the nature of his solution. His reference to 'extending it to other surfaces' refers to drawing maps on a surface other than a globe (or sphere). For example, suppose that we lived on a world shaped like the surface of an inner tube or ring doughnut – how many colours would we need then? (Mathematicians call such a surface a *torus*.) From his unpublished notes at Harvard, we know that Peirce found a torus map that needed six colours, but in fact we can do even better than this. The following torus map turns out to have seven mutually neighbouring countries and so requires seven

colours (we shall return to the colouring of maps on a torus in Chapter 7):

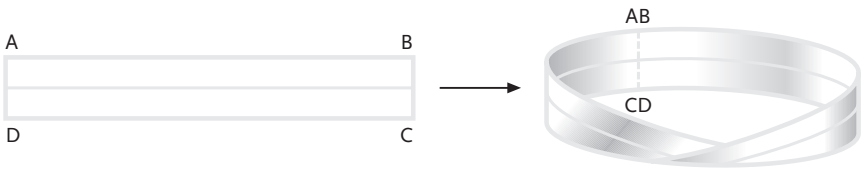
Peirce later remarked that the four-colour problem had been useful to him in testing the growth of his logical powers. Indeed, his researches into mathematical logic included the development of a 'logic of relatives', and in October 1869 he specifically applied this to map colouring (his approach is outlined in Chapter 5).

In June 1870, at the beginning of an extensive tour of Europe, Peirce visited the ailing De Morgan in London; it would be fascinating to know whether they discussed the four-colour problem. But by this time, the problem seems to have been largely forgotten in England: certainly, there is no evidence that the recipients of De Morgan's letters were any more interested in the problem than Hamilton had been on first hearing of it in 1852. On 18 March 1871, Augustus De Morgan died in London, having made little progress with the four-colour problem and unaware that over a century would elapse before a solution was discovered.

MÖBIUS AND THE FIVE PRINCES

As we have seen, the four-colour problem was originated by Francis Guthrie in 1852. However, it has sometimes been claimed, incorrectly, that the problem is older than this, dating back to a lecture given by the German mathematician and astronomer August Ferdinand Möbius around 1840. The *problem of the five princes* that Möbius posed is superficially similar to the four-colour problem, and we shall see how they came to be confused.

For many years, Möbius was professor of astronomy in Leipzig and director of the Leipzig observatory. In mathematics, his name is associated with the *Möbius function* in the theory of numbers and with *Möbius transformations* in geometry. But he is best remembered for the *Möbius strip*, or *Möbius band*, a curious object constructed from a long rectangular strip of paper by twisting one end through 180° and then glueing the two ends together, as pictured below. The resulting object has just one side and just one edge: this means that an ant could travel from any point on it to any other point without leaving the surface or going over the boundary edge. It was described by the sixty-eight-year-old



Möbius in late 1858, though it had already been constructed six months earlier by a professor of optics named Johann Benedict Listing, who has received little recognition for his prior discovery.

In one of his lectures on geometry, Möbius asked the following

August Ferdinand Möbius (1790–1868)

question, which had apparently been suggested to him by a Leipzig University friend, the philologist Benjamin Gotthold Weiske, who was greatly interested in mathematics:

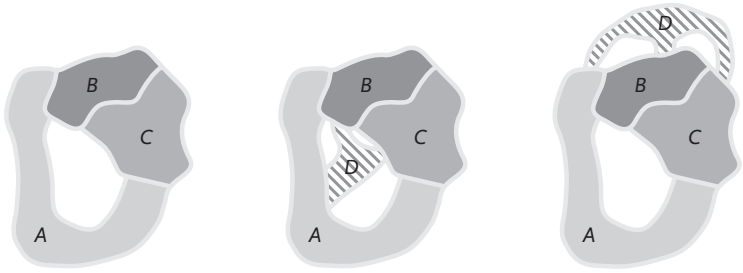
Problem of the five princes

There was once a king in India who had a large kingdom and five sons. In his last will, the king said that after his death the sons should divide the kingdom among themselves in such a way that the region belonging to each son should have a borderline (not just a point) in common with the remaining four regions. How should the kingdom be divided?

In the next lecture, Möbius's students admitted that they had tried hard to solve the problem, but without success. Möbius laughed and said he was sorry that they had struggled in vain, as such a division of the kingdom is impossible.

It is easy to see intuitively why Möbius's problem has no solution. Suppose that the regions belonging to the first three sons are called *A*, *B* and *C*. These three regions must all have boundaries in common with one another, as shown opposite in figure (a). The region *D* belonging to the fourth son must now lie completely within the area covered by the regions *A*, *B* and *C*, or completely outside it: these two situations are shown in figures (b) and (c). In each of these situations, it is then impossible to place

the region *E* belonging to the fifth son so as to have boundaries with the other four regions, *A*, *B*, *C* and *D*:



(a) (b) (c)

Möbius's problem of the five princes was later extended by Heinrich Tietze, who posed the following related question:

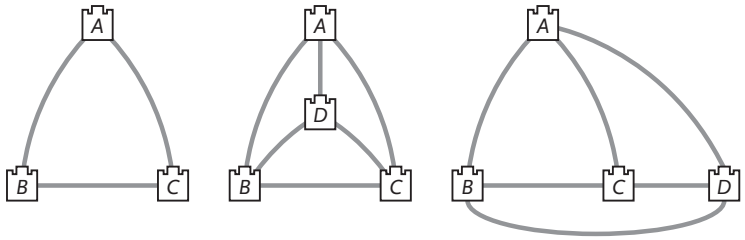
Problem of the five palaces

The king additionally required that each of his five sons should build a palace in his region, and that they should link the five palaces in pairs by roads in such a way that no two roads cross. How should the roads be placed?

This problem also has no solution. We can see why, by imitating the above explanation of the impossibility of solving Möbius's problem.

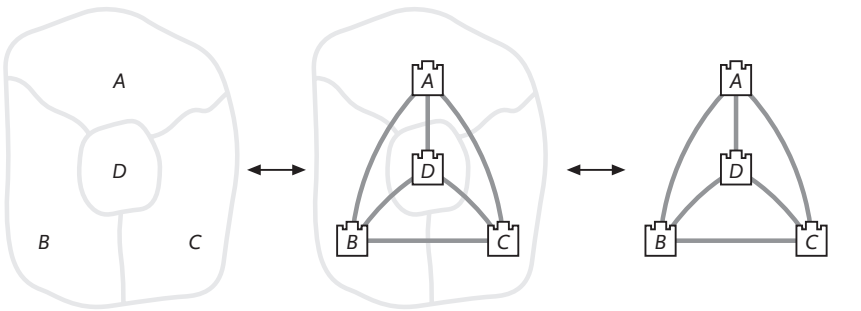
Suppose that the palaces belonging to the first three sons are called *A*, *B* and *C*. These three palaces can be linked by non-crossing roads, as shown in figure (a) below. The palace *D* belonging to the fourth son must now lie completely within the area

enclosed by the roads linking *A*, *B* and *C*, or completely outside it: these two situations are shown in figures (b) and (c). In each of these situations, it is then impossible to build the palace *E* belonging to the fifth son so as to link it by non-crossing roads to the other four palaces *A*, *B*, *C* and *D*.



(a) (b) (c)

Notice that the solution to either of these problems would have given a solution to the other: if the princes had been able to divide the kingdom into five mutually neighbouring regions, then they would also have been able to build palaces in the regions and construct non-crossing roads joining them. On the other hand, if the princes had been able to build the palaces and the roads joining them, then they would have been able to surround these palaces



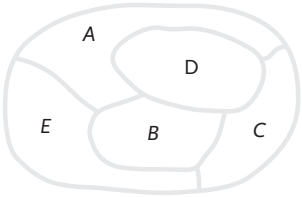
four neighbouring regions superimpose diagrams four interconnected palaces

by five neighbouring regions. Also, if the king had produced only *four* sons, then the kingdom could easily have been divided, as shown above, and the palaces could have been built and linked by non-crossing roads: with four sons, a solution to either problem yields a solution to the other.

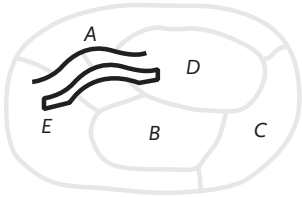
Before leaving Möbius's problem of the five princes, we should note that Heinrich Tietze gave a 'solution' to it. His description continued:

> The five brothers sank into despair as it became clear that it was not possible to fulfil the condition of their father's will. Suddenly a wandering wizard, who claimed to possess a solution, was announced . . . We can assume that the wizard was richly rewarded.

The wizard's solution was to connect two of the five regions by a bridge:

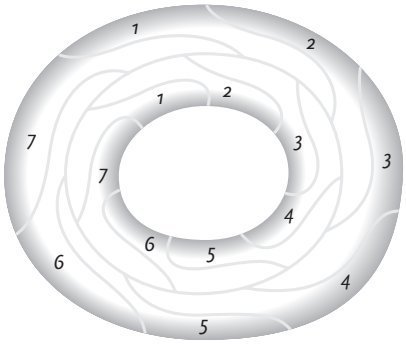


D and *E* not connected



D and *E* joined by a bridge

Of course, this is cheating, since we had assumed that we were restricted to drawing the kingdom on a plane, whereas the wizard's solution corresponds to solving Möbius's problem on the surface of a torus:



In fact, for such a problem the king could actually have had up to seven sons: the illustration above shows how to divide a torus into seven neighbouring regions, one for each of the seven sons. But if we are restricted to the plane, then, as we have seen, the maximum number of neighbouring regions is four – in the plane, five mutually neighbouring regions cannot exist.

CONFUSION REIGNS

What is the connection between Möbius's problem of the five princes and the four-colour problem, and why were they confused with each other? Before we answer this, let's have a cup of tea and sort out some logic! I can truthfully say that

if the tea is too hot, then I cannot drink it.

Another way of expressing this is to turn it round, and say that

if I can drink the tea, then it is not too hot.

What I cannot say is that

if the tea is not too hot, then I can drink it,

because there may be many other reasons why I cannot drink it: it may be too strong, or too sweet, or have a dead fly in it.

An arithmetical example of this kind of logic, involving the divisibility of whole numbers, is this:

if a whole number ends with 0, then it is divisible by 5.

For example, the numbers 10, 70 and 530 all end with 0, and all are divisible by 5.

Turning this round, we can deduce that

if a whole number is not divisible by 5, then it cannot end with 0.

For example, 11, 69 and 534 are not divisible by 5, and none of them ends with 0.

But we cannot say that

if a whole number is divisible by 5, then it ends with 0,

because there are many numbers, such as 15, 75 and 535, that are divisible by 5 but do not end with 0.

Logicians like to express such statements using symbols. If we use the letter *P* to stand for 'the tea is too hot' or 'a whole number ends with 0', and the letter *Q* to stand for 'I cannot drink it' or 'it is divisible by 5', we can put the first statement in each of the above chains of reasoning into this form:

if *P* is true, then *Q* is true – or, equivalently, *P* implies *Q*.

Turning this round, we get:

if *Q* is false, then *P* is false – or not-*Q* implies not-*P*.

But what we cannot say is:

if *P* is false, then *Q* is false – or not-*P* implies not-*Q*.

Let us now return to Möbius's problem of the five princes. Suppose that the terms of the king's will could be satisfied. Then

the region belonging to each of the five sons would share borderlines with the regions belonging to the other four sons – that is, there would be five neighbouring regions, each bordering the other four. If we wanted to colour these five neighbouring regions with different colours, we would need five colours (one for each region). So the four-colour theorem would be false.

The above argument tells us that

> if there is a map with five neighbouring regions, then the four-colour theorem is false.

(Here, P is the statement 'there is a map with five neighbouring regions', and Q is the statement 'the four-colour theorem is false'.)

Turning this around, as before, we find that

> if the four-colour theorem is true, then there is no map with five neighbouring regions.

What we cannot say is that

> if there is no map with five neighbouring regions, then the four-colour theorem is true.

So our little argument above, showing the impossibility of solving Möbius's problem, does not prove the four-colour theorem.

Over the years many people have attempted to prove the four-colour theorem by showing that no map can have five neighbouring regions. But as we have just seen, this does *not* prove the required result: their logic is the wrong way round.

One unwary person who fell headlong into this trap was the German geometer Richard Baltzer. On 12 January 1885 he gave a lecture to the Leipzig Scientific Society in which he described the problem of the five princes (which he had discovered among Möbius's surviving papers) and then explained why there cannot be five neighbouring regions. Baltzer published the results from

his lecture, wrongly claiming that the four-colour theorem follows immediately from his proof.

Baltzer's published paper was read by Isabel Maddison of Bryn Mawr College, in Philadelphia. In 1897 she wrote a 'Note on the history of the map-colouring problem' in the widely read *American Mathematical Monthly*, mentioning Baltzer's paper and remarking that 'it does not seem to be generally known that Möbius described the question, in a slightly different form, in his lectures in 1840'.

From there, the belief that Möbius was the first to formulate the four-colour problem spread far and wide, and was given credence when various well-known mathematics books, such as Eric Temple Bell's *The Development of Mathematics*, repeated the error. It was not until 1959 that the geometer H. S. M. Coxeter set the story straight, and since then Francis Guthrie has been universally recognized as the true originator of the four-colour problem.

We shall resume our historical development of the four-colour problem in Chapter 4, but first we head back to the eighteenth century to investigate the world of polyhedra.