MacColl's influences on Peirce and Schröder

Irving H. Anellis
Peirce Edition, Institute for American Thought
Indiana University-Purdue University at Indianapolis, USA

Résumé : Les contributions à la logique de MacColl et Charles Sanders Peirce (1839-1914) ont été les deux plus profondes influences sur le travail de Ernst Schröder (1841-1902) en logique algébrique. Dans son *Vorlesungen über die Algebra der Logik,* Schröder a cité MacColl comme l'un de ses précurseurs les plus importants. Schröder a comparé les travaux de Peirce avec les premières parties de la série d'articles intitulés « The calculus of equivalent statements » que MacColl publie entre 1877 et 1880. Schröder a attribué la priorité à MacColl pour avoir anticipé les résultats de Peirce. Pour Schröder, MacColl était une phase préliminaire à l'algèbre de la logique de Peirce.

Abstract: The contributions to logic of MacColl and Charles Sanders Peirce (1839-1914) were the two most profound influences upon the work of Ernst Schröder (1841-1902) in algebraic logic. In his *Vorlesungen über die Algebra der Logik,* Schröder referred to MacColl as one of his most important precursors. Schröder compared Peirce's considerations with the early parts of MacColl's series of papers "The calculus of equivalent statements" (published between 1877 and 1880), and he attributed to MacColl priority for having anticipated Peirce's results. For Schröder, MacColl's calculus was a preliminary stage of Peirce's algebra of logic.

Introduction

Hugh MacColl (1837-1909) was originally a mathematician, who gradually turned towards logic. He contributed to propositional logic, to nonclassical logic, especially modal logic, and the theory of inference. He suggested propositions and implication as the basis for logic, being among the first, if not *the* first, to articulate the modern conception of implication, introducing the terms "implication" and "non-implication" in the second installment of "The Calculus of equivalent statements" [MacColl 1877-1878b, 181]. Towards the end of his career, he engaged in debate with Bertrand Russell (1872-1970) in the pages of *Mind* on the existential import of propositions. In his review of MacColl's *Symbolic Logic* [Russell 1906], Russell wrote:

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> He is primarily concerned with *implication*, not with *inclusion*; his formulæ state that one statement implies another, not (directly) that one class is contained in another. The relation of inclusion between classes is for him derivative, being in fact the relation of implication between the statement that a thing belongs to the one class, and the statement that it belongs to the other. He was, I believe, the first to found symbolic logic on propositions and implication, and in this respect he seems to me to have made an important advance upon his predecessors. [Russell 1906, 255]

And in his manuscript on "Recent Italian work on the foundations of mathematics" (1901), Russell, contrasting the conception of the algebraic logicians with that of MacColl and Gottlob Frege (1848-1925), wrote that:

> Formal Logic is concerned in the main and primarily with the relation of implication between propositions. What this relation is, it is impossible to define: in all accounts of Peano's logic it appears as one of his indefinables. It has been one of the bad effects of the analogy with ordinary Algebra that most formal logicians (with the exception of Frege and Mr. MacColl) have shown more interest in logical equations than in implication. [Russell 1993, 353]

MacColl carried out his work in isolation and independence from the leading logicians of his day. Christine Ladd-Franklin (1847-1930), a former student of Charles Sanders Peirce (1839-1914), remarked in "On some characteristics of symbolic logic," for example, that "Nothing is stranger in the recent history of Logic in England, than the non-recognition which has befallen the writings of the author [H. MacColl]" [Ladd-Franklin 1889, 562]. This does not mean that his work was entirely unknown, however. There were references to his work by a few leading logicians; Giuseppe Peano (1858-1932) in his *Arithmetices Principia, Nova Methodo Exposita* [Peano 1889, iv*n*] and *Formulaire de mathématique* acknowledged a debt to MacColl's work in propositional logic, in particular employing results [Peano 1901, v, 13–14, 16, 19–21, 226] from MacColl's "Calculus of equivalent statements" [MacColl 1877-1878a, 1878-1879]. This led Russell to write that "Peano, like McColl, at first regarded propositions as more fundamental than classes..." [Russell 1903, 13], and Louis Couturat (1868-1914) worked on his propositional logic, without, however, taking account of either MacColl's modal logic or probability logic [Couturat 1899, 621].

MacColl is indubitably best known to historians and philosophers of logic for his exchanges with Bertrand Russell in the pages of *Mind* on the problem of the existential import of propositions. He has also received attention from those interested in the history of nonclassical logics because of his criticisms, related to the question of existential import, of the accepted definition of material implication. Norbert Wiener (1894-1964), for example, had strongly praised Clarence Irving Lewis (1883-1964) for the courage to challenge in print

Russell's use of material implication in *Principia Mathematica* [Wiener 1916, 656], while Lewis himself criticized all logicians for adhering to the classical conception of implication as material implication, while explicitly exempting only MacColl [Lewis 1912]. [1] MacColl's definition of the syllogism as an inference defined in terms of implication, however, was perhaps his more significant contribution to the over-all development of logic. This translation of the syllogism as an implication was a crucial step in establishing propositional logic in its modern form, and was adopted by such widely scattered logicians as Peirce and Russell.

Whereas George Boole (1815-1864) had tended to limit the interpretation of his logic to the calculus of classes, MacColl's first contribution to logic was to admit not only a class interpretation, but a propositional interpretation. He indeed gave preference to the propositional interpretation because of its generality, and called it "pure logic." Most of MacColl's work proceeded in an extensive series of journal articles. But in *Symbolic Logic and its Applications* (1906), he published the final version of his logic(s). There, propositions are qualified as either certain, impossible, contingent, true or false.

The contributions to logic of MacColl and Peirce were the two most profound influences upon the work on Ernst Schröder (1841-1902) in algebraic logic, and in his *Vorlesungen über die Algebra der Logik* [Schröder 1890-1905]. Volker Peckhaus, who counted the number of references to Peirce and MacColl in the *Vorlesungen,* noted that, whereas Peirce was cited most often, MacColl was the second most cited author [Peckhaus 1998, 20]. So far as Schröder was concerned, MacColl's calculus held, in Peckhaus's words, "the status of a preliminary stage of Peirce's algebra of logic" [Peckhaus 1998, 20].

Peckhaus has already dealt in substantial detail with Schröder's views of MacColl, and of MacColl's impact upon Schröder's work in logic [Peckhaus 1998]. I shall, therefore, concentrate my discussion on Peirce's views of MacColl, and of the influences which MacColl's work exerted upon Peirce. It is, however, worth repeating, as [Peckhaus 2004, 584] also noted, that Schröder explicitly expressed his agreement with John Venn (1834-1923) regarding the close similarity between Peirce's and Schröder's work and MacColl's.

MacColl's influences on Schröder

Schröder's own techniques were primarily modifications of Peirce's, and the logical system of the *Algebra der Logik* can be understood as a unification, systematization, and development of Peirce's logic and an extension of

1. [Read 1998, 59–60] notes that for the later edition of his *Survey,* Lewis [Lewis 1960] excised nearly all of the material discussing his early work on systems of strict implication, which included the early references to the work of MacColl and remarking on his indebtedness to MacColl. He argued by way of justification for the wholesale abridgment that the early Lewis-Langford logic of strict implication had been superseded.

Peirce's algebra of relatives—and, Schröder would also claim, improvement, of Peirce's logic. Schröder, Peckhaus reminds us, "always compared Peirce's considerations with the early parts of MacColl's series of papers "The Calculus of Equivalent Statements" published between 1877 and 1880, and he conceded to MacColl priority when he had anticipated Peirce's results" [Peckhaus 1998, 20]. Taking this seriously, we must examine the extant evidence to determine the extent to which MacColl is to be understood as a precursor of Peirce and the nature of the influences which MacColl's work exerted on Peirce's own work in logic.

Peckhaus identifies three aspects in which Schröder was influenced by MacColl, either directly or through Peirce [Peckhaus 1998, 20–21]. (1) In investigating the comparative abilities of various calculi to eliminate terms and resolve logical formulas, Schröder compared his own procedure, adapted from Boole's, with Peirce's. Peirce's was taken as the natural way to operate, and MacColl's method, which Schröder investigated *in extenso,* was considered to be a "preliminary stage" of Peirce's. (2) Schröder explicitly acknowledged MacColl's priority in formulating propositional logic, terming it the "MacColl-Peircean propositional logic." Schröder was, however, critical of their purported efforts to base logic upon the calculus of propositions, arguing that his founding of the propositional calculus upon the calculi of domains (*Denkbereiche*) and classes was more general. It must be noted that Schröder's criticism applies to Peirce only to the extent that he undertook to generalize Boole's efforts to formulate the categorical syllogisms of Aristotle as chains of propositions whose fundamental connective was implication. But once Peirce undertook to treat his primary connective, *illation,* the so-called "claw," as interpretable as any transitive, reflexive, and asymmetric relation, holding between either terms, propositions, relations, classes, or sets, depending upon the context in which the formulas occurred, this criticism could no longer, strictly, apply. For quantified formulas, only the Boolean part of formulas could be treated strictly as propositions. (3) Awarding priority to MacColl and Peirce for defining material implication, Schröder adopted that definition for his own propositional calculus. Setting aside the question of priority, Schröder after proving his exposition of MacColl's work [Schröder 1890-1905, I, 589–592], expresses his agreement with John Venn that MacColl's method is "practically identical with those of Peirce and Schröder" [Venn 1881, 414; 1894, 492]. To appreciate Schröder's positive attitude towards MacColl, it is perhaps sufficient to recall Schröder's testimony, in the first volume of the *Vorlesungen über die Algebra der Logik* [Schröder 1890-1905, I, 560] that he accounted only MacColl's and Peirce's methods equal to his own in their proficiency. The advantage that Schröder identified in the logical systems of Peirce and MacColl over those of Venn and William Stanley Jevons (1835-1882), is the greater naturalness and simplicity of the former [Schröder 1890-1905, I, 573]. His quibble with the "MacColl-Peircean propositional logic" was its lengthiness, and he undertook in the pages that followed [Schröder 1890-1905, I, 574–584] to repair that defect by modifying Peirce's methods as presented in Peirce's

"On the algebra of logic" [Peirce 1880, 37–42]. Schröder's changing attitude towards MacColl's work is summed up by Shahid Rahman and Juan Redmond as follows: "Ernst Schröder, who in his famous *Vorlesungen über die Algebra der Logik* quotes and extensively discusses MacColl's contributions, first had quite a negative impression of MacColl's innovations though later he seems to have changed his mind, conceding that MacColl's algebra has a higher degree of generality and simplicity in particular in contexts of applied logic. What Schröder definitively rejects is MacColl's propositional interpretation of the Aristotelian Syllogistic" [Schröder 2008, 538]. Peckhaus [Peckhaus 1998, 30] is led to think that Schröder learned of MacColl's work indirectly, through references by Peirce, but notes as well that Schröder, treating MacColl primarily as historical, consistently acknowledges MacColl nonetheless as a predecessor of Peirce's and his own work. One of the principal and most severe failures of MacColl's system is the limitation imposed by his failure to introduce a fully fledged apparatus for quantifiers into the system, rendering his system far less flexible than Peirce's and Schröder's, and ensursing that it would remain a propositional logic only. [2]

MacColl's influences on Peirce

MacColl developed a propositional logic within a general Boolean framework in the second installment of his paper "The calculus of equivalent statements" [MacColl 1877-1878b], and provided an example and exposition of what he called "tabular logic" ("*logique tabulaire*") in his 1902 article of that title in the *Revue de métaphysique et de morale* [MacColl 1902a]. In his interpretation, as expressed in the opening of the first installment of "The calculus of equivalent statements," $A = 1$ means that the statement A is true; $A = 0$ that it is false [MacColl 1877-1878a, 9]. The second installment opens with the explanation that the expression $A : B$ means that statement A implies statement B, i.e. in any case if A is true, then B is true [MacColl 1877-1878b, 177]. In the third installment, MacColl enumerated what he considered to be the three principal differences between his development of logic and Boole's and Jevons's [MacColl 1878-1879, 27]. They are that (1) every letter, as well as every combination of letters, signifies a statement; (2) that he introduces the symbol ':' to denote that the statment following it is true, provided the statement preceding it is true; and (3) he introduces a special symbol for negation, which may be applied, depending upon circum-

2. Whether modal operators, with MacColl's 'quasi-modal' values of certain, impossible, contingent, true or false, could or should have been reconstituted as fully fledged quantifiers, in line with Russell's interpretation of universally quantified propositions as necessary and existentially quantified propositions as possible, is critical, but beyond the scope of our present concern, and it is, moreover, moot, since MacColl did not adopt that option; see [Anellis 2009, 154] and [Rahman & Redmond 2008, 543–544].

stances, to any number of terms, or even a complex statement. For our particular purposes, the second innovation is crucial for distinguishing the classical Boolean algebra from the symbolic logic of MacColl in its more significant aspect because it introduces implication as the principal connective of the system (along with negation), and can be seen in this sense as closely related to Schröder's subsumption [Peckhaus 2004, 584]. It differs significantly, however, from Peirce's (and Schröder's) introduction of implication into the algebra of logic, given that, as here defined by MacColl, it amounts to *strict implication,* whereas Peirce had adopted *material implication,* together with negation, as his main connectives.

Although remembered today primarily, when at all, for his discussions with Russell on the nature of propositions and their existential import—to the extent that Max Fisch and Atwell Turquette thought that Peirce might have been led to work out a triadic logic by the debate between MacColl and Russell on the nature of implication [Fisch & Turquette 1966, 82–83]—MacColl was in his day a respected member of the logical community, taking his position in the competition for the best (most effective) logical system. Schröder's comparison of the procedures for solving logical problems provided by different logical systems, in particular the different ways of solving Boole's famous "Example 5," is discussed. MacColl's work demonstrates the importance of the organon aspect of symbolic logic. We have:

> Ex. 5. Let the observation of a class of natural productions be supposed to have led to the following general results.
>
> 1^{st}, That in whichsoever of these productions the properties A and C are missing, the property E is found, together with one of the properties B and D, but not with both.
>
> 2^{nd}, That wherever the properties A and D are found while E is missing, the properties B and C will either both be found or both be missing.
>
> 3^{rd}, That wherever the property A is found in conjunction with either B or E, or both of them, there either the property C or the property D will be found, but not both of them. And conversely, wherever the property C or D is found singly, there the property A will be found in conjunction with either B or E, or both of them.
>
> Let it then be required to ascertain, first, what in any particular instance may be concluded from the ascertained presence of the property A, with reference to the properties B, C, and D; also whether any relations exist independently among the properties B, C, and D. Secondly, what may be concluded in like manner respecting the property B, and the properties A, C, and D. [Boole 1854, 146]

In an entry into his *Logic Notebook* for 14 November 1865 (see MS 114 of Peirce's *Logic Notebook* of 1865-1909, published in [Peirce 1982, 337]), Peirce first declared that there is "no difference logically between hypotheticals and categoricals," and that "[t]he subject, is a sign of the predicate, the antecedent of the consequent," thus virtually, if not yet actually, equating the copula of predication with implication and opening the way to eventually represent the syllogism "All S are M, All M are S, Therefore all S are P" as $(S \supset M)\&(M \supset P) \supset (S \supset P)$. This translation is, as we noted, strongly suggested in [Peirce 1870], and also, independently, ten years later, by Hugh MacColl in his 1880 paper on "Symbolical Reasoning," where he wrote in his Definition 3 that:

> The symbol :, which may be read "implies," asserts that *the statement following it must be true, provided the statement preceding it be true.*
>
> Thus, the expression $a : b$ may be read "a implies b," or "If a is true, b must be true," or "Whenever a is true, b is also true." [MacColl 1880a, 50–51]

MacColl followed this up in "Symbolical Reasoning IV," when he explains that:

> from two *implicational* premisses $A : B$ and $B : C$ draw *implicational* conclusion $A : C$. That is to say [...] $(A : B)(B : C) : (A : C)$. [MacColl 1902b, 367]

MacColl made this reading more fully explicit, in case it still needed to be by this time, where he wrote that, in the case, for example, of the *Barbara* syllogism "All A are B, All B are C, *therefore* All A are C,"

> In this form the syllogism is true whether premisses or conclusion be true or false [...] and must, therefore, be classed amongst the *formal certainties.* Now a statement is called a formal certainty when it follows necessarily from our formally stated conventions as to the meanings of the words or symbols which express it; and until a language has entered upon the propositional stages those conventions (or definitions) cannot be formally expressed and classified. [MacColl 1902b, 368]

For MacColl, there is no distinction between categorical and hypothetical propositions.[3] This is one of the essential features permitting MacColl, Peirce, and Venn, and later Bertrand Russell, to understand the syllogism of the form "All A are B. All B are C. Therefore all A are C" as the implication $[(A \supset B)(B \supset C)] \supset (A \supset C)$. Venn in *Symbolic Logic* objected to MacColl's notation regarding implication, and in particular the appellation "implication," however, principally on the ground that MacColl applies the term "implication" to hypotheticals; "implication," Venn argued, "implies" that it is known that there is a connection between the terms, and this is a stricter condition than the "if... then" relation [Venn 1881, 376–378]. Thus Venn's objection regards not the treatment of the logical relation which we call "implication," but rather his calling it by that name. Ladd-Franklin considered it one of MacColl's greatest

3. See, e.g. [Rahman 2000].

accomplishments to define Aristotle's *A*-proposition in terms of implication, writing that "The logic of the non-symmetrical affirmative copula, "all *a* is *b*," was first worked out by Mr. MacColl" [Ladd-Franklin 1889, 562]. She then explains that:

> Nothing is stranger, in the recent history of Logic in England, than the non-recognition which has befallen the writings of this author. The fact that his contributions appeared in a journal which logicians were not in the habit of referring to (his brief article in *Mind* did not do his method justice), the fact that he was not acquainted with the writings of Boole, and the further accident that he considered it a matter of importance to read "all *a* is *b*," which he wrote $a : b$, in the words, "the statement that a thing is *a* implies the statement that it is *b*,"—all doubtless contributed to making him seem foreign to writers trained in the usual schools of logic. The fact that the nature of the connection between the terms in "all *x* is *y*," and between the propositions in "*a* is *b* is-always-followed-by *c* is *d*," is exactly the same, and must be exhibited in the same formal rules of procedure, has nothing to do with the words in which the proposition and the sequence may be expressed. But if it had not been for this accidental misfortune, it seems incredible that English logicians should not have seen that the entire task accomplished by Boole has been accomplished by MacColl with far greater conciseness, simplicity and elegance; and, what is an interesting point, in terms of that copula which is of by far the most frequent use in daily life. [*sic*] [Ladd-Franklin 1889, 562]

MacColl's preference for multiple-valued, especially triadic logic over bivalent logic was the subject of criticism by Platon Sergeevich Poretskii (1846-1907), [4] and Jevons [Jevons 1881, 486] among those who challenged his definition of implication. Nevertheless, MacColl's contributions to non-classical logic were more consequential than his work in two-valued logic. His definition of inference in terms of implication, and his concept of strict implication, aimed at avoiding the *ex falso sequitur quodlibet* of the accepted definition of material implication which he himself had framed, is a precursor to the introduction by Clarence Irving Lewis (1883-1964) of strict implication [Lewis 1912; 1913].

Whether Peirce had MacColl in mind among those of whom he wrote that: "The algebra of logic was invented by the celebrated British mathematician George Boole, and has subsequently been improved by the labors of a number of writers in England, France, Germany, and America," [Peirce 2010, 63], can only be conjectural, since, in the manuscript "Boolian Algebra" of *circa* 1890 in which Peirce made this comment, no individual names, other than Boole's,

4. Styazhkin asserts, without supplying citations, that MacColl's "tendency toward augmenting two-valued logical formalism with a third truth value... evoked many objections from P. S. Poretskiy" [Styazhkin 1969, 214].

are mentioned.[5] Neither does he clearly make explicit in this context the specific improvements that he had in mind, perhaps because the manuscript in question was intended to serve as an introductory logic textbook rather than as either a critical or an historical exposition. Instead, we must go to the manuscript "Boolian algebra—elementary explanations" [Peirce 2000, 5] of 1886 to achieve an explicit reference by Peirce to MacColl's work. There he compares his so-called "claw" of illation as roughly equivalent to MacColl's representation, as introduced by MacColl in [MacColl 1877-1878b, 177], of the connective of implication expressing the relation between antecedent and consequent. Peirce writes:

> In my different publications I have used a sign like Y turned over on its side, $\prec$, to signify the relation between the antecedent and consequent of an hypothetical proposition. It is a very convenient sign, but as I have no such sign on this typewriter, I shall use a colon for the same purpose, according to the practice of Mr. Hugh MacColl. [Peirce 2000, 5]

Peirce had already referred to MacColl's work in logic a decade earlier, in his paper "On the algebra of logic" of 1880. This was, in fact, Peirce's earliest published reference to MacColl, and it is again, specifically in connection with MacColl's use in [MacColl 1877-1878b, 183] of the sign for implication several times over in a single proposition. Peirce stresses the role of implication in connection precisely to MacColl's Def. 12 [MacColl 1877-1878b, 177]. The reference to MacColl there occurs in a footnote [Peirce 1880, 24], in which Peirce writes: "Mr. Hugh McColl (*Calculus of Equivalent Statements,* Second Paper, 1878, 183) makes use of the sign of inclusion several times in the same proposition. He does not, however, give any of the formulæ of this section," concluding Peirce's explanation for the use of his illation in defining the categorical propositions A, E, I, and O, their Aristotelian transformations, and the determination of the truth-values of the basic four forms of categorical propositions. In the same paper, Peirce notes that four different algebraic methods had been devised for solving problems in the logic of non-relative terms, namely those of Boole, of Jevons, of Schröder, and of MacColl, to which he himself added a fifth [Peirce 1880, 37]. Peirce nevertheless finds in a footnote that MacColl, "apparently having known nothing of logical algebra except from a jejune account of Boole's work in Bain's *Logic* [1870] published several papers on a *Calculus of Equivalent Statements*" [Peirce 1880, 32].[6]

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5. We do not know for certain whether Peirce would have accounted MacColl among the British, the French, or British residing in France. In any event, since Peirce here gives no names, he could possibly have had in mind either Louis Liard (1846-1917), whose work included studies of Jevons' and Boole's contribution to logic [Liard 1877a; 1877b], as well as a wider survey of British work in algebra of logic [Liard 1878], or the Belgian Joseph-Rémi Léopold Delboeuf (1831-1896), author of "Logique algorithmique" [Delboeuf 1876].

6. The footnote in question runs half a page long. Only the final two sentences refer to MacColl, and explicitly to [MacColl 1877-1878b]. In that note, Peirce wrote in

In fact, MacColl testified in the second installment of his "The calculus of equivalent statements" [MacColl 1877-1878b, 178] that he had not read Boole's work. Peirce in a footnote likewise noted that, as a consequence of his unfamiliarity with the latest work in algebraic logic beyond that gleaned from the sophomoric account by Alexander Bain (1818-1903), MacColl's contributions to logic there was based upon "nothing but the Boolian algebra, with Jevons's addition and a sign of inclusion" [Peirce 1880, 32]. It was not until after his "Bainsian" treatment of Boole's logic that MacColl became directly acquainted with Boole's work, and in particular with Boole's *Laws of Thought* (1854). This came about through the good offices of Robert Harley (1828-1910), who loaned him a copy of that work [MacColl 1878-1879, 27]. [7]

Schröder, for his part, suggested that the five methods referred to by Peirce could, in fact, be reduced to three, because his own was a modification of Boole's, which thereby became obsolesent [Schröder 1890-1905, I, 559], while Peirce's and MacColl's were essentially sufficiently similar to permit their reduction to, or combination into, one, the "MacColl-Peircean" [Schröder 1890-1905, I, 589]. MacColl, on the other hand, claimed the superiority of his methods over that of the "Boolian Logicians" for solving certain logical and mathematical problems, specifically those that involve questions of probability, and he indicated that his logic was developed explicitly in response to questions about probability [Rahman & Redmond 2008, 556]. [8]

The explanations for MacColl's lack of direct familiarity with the work of Boole and the other algebraic logicians, and for the relatively insubstantial knowledge which he had of Boole's logic were, first, his comparative isolation from the small circle of investigators in algebraic logic, [9] his limited linguistic knowledge, on account of which he was unable to read much of the work of his contemporaries and immediate predecessors [Grattan-Guinness 1998, 10], and his insubstantial mathematical background relative to that shared by the majority of algebraic logicians of that period, achieving only a B. A. in mathematics from the University of London in 1876. Despite this, and in particular, we

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full: "Later in the same year, Mr. Hugh McColl, apparently having known nothing of logical algebra except from a jejune account of Boole's work in Bain's Logic, published several papers on a *Calculus of Equivalent Statements,* the basis of which is nothing but the Boolian algebra, with Jevons's addition and a sign of inclusion. Mr. McColl adds an exceedingly ingenious application of this algebra to the transformation of definite integrals" [Peirce 1880, 32].

7. Interestingly, and perhaps not merely coincidently, Harley performed a similar service for Jevons, by reporting [Jevons 1877, xxi] that Leibniz had anticipated Boole's laws of logical notation, and noting additionally that Boole had been informed about Leibniz's work by Robert Leslie Ellis (1817-1859) a year after Boole published his *Laws of Thought.*

8. See [MacColl 1881] for his negative appraisal of Peirce's conception of probability.

9. Thus, in a letter of 1905 to Bertrand Russell, MacColl complained: "I feel myself an Ishmael among logicians, with my hand against every man, and every man's hand against me" (quoted in [Astroh 1998, 142]).

might add, despite MacColl's scanty mathematical background, Peirce judged, indirectly referring to [MacColl 1877-1878a] that "Mr. McColl [*sic*] adds an exceedingly ingenious application of the algebra to the transformation of definite integrals" [Peirce 1880, 32]. MacColl devised his techniques so as to apply his logical calculus to the theory of limits in an effort to expand and develop the logical system as a proper analog of algebra. Specifically, as [Grattan-Guinness 1998, 7] noted, referring to 35 pages of his 1906 *Symbolic Logic*, MacColl sought to establish the analogy between truth-values in compound propositions and the laws of signs in algebra. For example, the conjunction True and False is False in logic, and positive times negative is negative in algebra. MacColl, as [Grattan-Guinness 1998, 7] also notes, used superscripts, with "P" and "N" to define appropriate propositions such as

$$(x-3)^P = (x>3), \text{and } (x-3)^N = (x<3)$$

with "=" serving as equality by definition—in addition to its use as arithmetical equality and equivalence between propositions [MacColl 1906, 107]. But, while change of limits in multiple integrals, where inequalities such as in (1) were used to state that a variable lay between given values, and as Peirce said, MacColl's procedures were "exceedingly ingenious"; but for some of the more elementary needs, such as obtaining the roots of a quadratic equation [MacColl 1906, 112–113], were far more complicated than necessary.[10]

MacColl's *variables,* as presented in the fifth installment of "The calculus of equivalent statements" [MacColl 1896, 157), are similar in several respects to Peirce's assertion of possibility. In the more mature development of his theory of modal propositional logic, as found in the sixth installment of his

10. The fact however that MacColl's innovations with integrals on this score actually are less efficient than the standard procedures, and the confusions which he exhibited with regard to questions of defining the use of zero in expressions involving differentials as ratios df/dx between a function f and its respective variable x, indicate that, whereas MacColl was well within the French tradition of Lagrange of treating the theory of limits algebraically, he was not familiar with the rigorous Cauchy-Weierstrassian reformulation of the theory of limits in terms of $\varepsilon - \delta$ notation, in which a careful distinction is made between [absolute] zero and a very small ε approaching, but not attaining zero, in relation to the change, from either the positive or negative side, of δ where the limit of $f(x)$ as x approaches a is L, i.e. $\lim_{x\to a} f(x) = L$ if for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that $| f(x) - L | < \varepsilon$ whenever $0 < | x - a | < \delta$, despite Weierstrass's students helping to disseminate this across Europe from the 1870s forward.
Grattan-Guinness also takes note of MacColl's weakness in the history of mathematics, in this case again with respect to the meaning of dy/dx; Grattan-Guinness tells us that in a letter that MacColl adressed to Russell on 18 December 1909 summarizing "an argument about limits in a paper published posthumously in *Mind* [MacColl 1910], he wondered why the notation "dy/dx" could not be read as a ratio of infinitesimals [...], apparently unaware that this was *exactly* how Leibniz introduced it (with his own sense of infinitesimal) and his successors used it [Bos 1974]." [Grattan-Guinness 1998, 10].

"Symbolic reasoning," these variables became for MacColl existentially defined assertions; there MacColl wrote that:

> Firstly, when any symbol A denotes an *individual*; then, any intelligible statement $\phi(A)$, containing the symbol A, implies that the individual represented by A has a *symbolic* existence; but whether the statement $\phi(A)$ implies that the individual represented by A has a *real* existence depends upon the context. Secondly, when a symbol A denotes a *class,* then the intelligible statement $\phi(A)$ containing any symbol A implies that the whole class A has a *symbolic* existence; but whether the statement $\phi(A)$ implies that the class A is *wholly real,* or *wholly unreal,* or *partly real and partly unreal,* depends upon the context. [MacColl 1905, 77]

Bain's conception of Boole's principal contribution to the syllogistic logic is that, with the addition of the axiom that equals added to equals yield equal sums, and that, if A is greater than B and B is greater than C, then A is greater than C, he "draws the Syllogism under the axiom that suffices for the reduction of equations," he "assumes that the analogy of the logical method and the algebraical is sufficiently close to allow of the substitution" [Bain 1870, 164]. Boole's additions to the syllogism, then consist in drawing the details of the algebraic representation of logical propositions; these are adumbrated by Bain in a few lines short of seventeen pages [Bain 1870, 190–207].

Examining the details of MacColl's major productions, we find that the first two papers on "The calculus of equivalent statements" [MacColl 1877-1878a-b] and his first paper on "Symbolical reasoning" [MacColl 1880a] present a calculus of propositions which has essentially the properties of Peirce's, albeit without Π and Σ operators to serve, as they did for Peirce (and then Schröder), as logical product and logical sum, doing duty for the universal and existential quantifiers respectively. It is, therefore, in fact a calculus of propositions, like the Boolean (two-valued) algebra of logic with which we are familiar. Lewis [Lewis 1918, 108] believed that the date of these papers indicates that MacColl arrived at their content independently of Peirce's work along the same lines.

In his *Symbolic Logic and its Applications* (1906), MacColl's attention is directed to truth-functional logic, and hence the fundamental symbols represent propositional functions rather than propositions. It is also here that MacColl examines a trivalent logic. Instead of just the two traditional truth values *true* and *false,* we also have *certain, impossible* and *variable* (not certain and not impossible). These are indicated by the exponents $\tau, \iota, \varepsilon, \eta$ and θ respectively. The result is a highly complex system, the fundamental ideas and procedures of which suggest somewhat Lewis's system of strict implication to be set forth in Chapter V of his *Survey of Symbolic Logic* (1918). The logical theory of MacColl's *Symbolic Logic* is neither mathematically nor logically equivalent to the Boole-Schröder calculus. We will return to a consideration of the question of the relation between Peirce's trivalent logic and MacColl's four-valued, "three-dimensional" logic.

One of the questions of priority which would appear to be problematic is whether Peirce or MacColl first proclaimed that Boole's calculus was amenable to alternative interpretations. It is well known that Peirce allowed his "claw" of illation to function either as material implication or as class inclusion. Indeed, one of the criticisms which both Peano and Russell leveled against the Boole-Schröder calculus was that it neither offered nor provided a basis for distinguishing between classes and propositions. Russell's criticism of the equivocal use by Peirce and Schröder of one connective for both inclusion and implication occurred first in his "Sur la logique des relations avec des applications à la théorie des séries" [Russell 1901], and recurs, in somewhat different guise, in the *Principles of Mathematics* [Russell 1903, 24]. Frege's similar criticsm of Schröder occurs in his review [Frege 1895] of Schröder's *Vorlesungen über die Algebra der Logic.* Likewise, in his *Arithmetices Principia, Nova Methodo Exposita* [Peano 1889, V], Peano laid stress on the difference between set elementhood (or class membership) on the one hand and material implication on the other. From the perspective of Russell and Peano, it accrues to the disadvantage of MacColl that, although he placed his greater emphasis on the logic of propositions, he nevertheless allowed for the interpretation of his calculus both with respect to propositional logic and the class calculus. By the same token, it accrued to his advantage, from the perspective of Peirce and Schröder, that he imposed no constraints between these interpretations. For Peirce's illation and Schröder's equivalent Subsumption ($\Subset$), they more probably initially had in mind for this connective general inclusion, that is, any relation that is transitive, reflexive, and asymmetrical, rather than material implication specifically [Houser 1987, 431, 437 n. 6]. However, its interpretation, as either inclusion or implication, could vary, according to the specific context in which it occurred. The potential for multiple interpretations of the kind made by Peirce for his illation and Schröder's subsumption could be justified, prior to Cantor's development of set theory by the pre-Cantorian treatment of collections, in terms of the part-whole relationship since it is only after the development of the concepts of set membership and class inclusion and the consequent distinction between the two notions that it made sense to distinguish between these two kinds of relation or that such a distinction had a purpose.[11] MacColl was among those who, like Peano and Russell, criticized Peirce's illation and Schröder's Subsumption for conflating implication and inclusion [MacColl 1906, 78]. But this is reflective of MacColl's inability to appreciate that algebraic logic is concerned with the general algebraic structure of equations as relations, rather than with propositional logic specifically, which was MacColl's concern. Whether, however, MacColl's myopia in this regard is rooted in his comparative isolation from the mathematical community working in algebraic logic, or something else, is perhaps moot. What is salient

11. For a philosophical approach to the history (and "pre-history") of set theory and the difference between the Aristotle-inspired conception of infinity, based on the whole-part relation, and concept of infinity in Cantorian set theory as based upon the element-set relation, see, e.g. [Tiles 1989].

is that this is indicative of the primary difference between the "Booleans" on the one hand and MacColl on the other.

Nathan Houser [Houser 1989, l] has noted that Peirce reported on 14 December 1880 that he had received copies of papers from a number of authors "on various logical and psychological subjects," and that among those whose works he received was MacColl. What Houser does not specify is which of MacColl's writings he had received, or when, but we may certainly conclude that Peirce must at the very least have known enough about MacColl's "The calculus of equivalent statements" by 1880 to permit his reference to it in his own "On the algebra of logic" and explicit citation of the second installment of that series. Houser [Houser 1989, lix] also notes that "Peirce had made plans to visit with Hugh MacColl in Boulogne-sur-Mer, which he probably did near the beginning of his stay," which would indicate the occurrence of a personal meeting between Peirce and MacColl some time in mid-May 1883 [MacColl 1883].

In 1883, *Studies in Logic* [Peirce 1883] appeared, edited by Peirce and containing contributions by Peirce and his Johns Hopkins University students. In the "Preface," Peirce noted that between 1864 and 1877, Boole in the *Laws of Thought,* Jevons in *Pure Logic* [Jevons 1864], Robert Grassmann (1815-1901) in his *Die Formenlehre oder Mathematik* [Grassmann 1872a], and, presumably more explicitly, in his *Begriffslehre oder Logik* [Grassmann 1872b], Schröder in his *Operationskreis* [Schröder 1877], MacColl in the "The calculus of equivalent statements," and Venn all, independently of one another, proposed the use of the addition sign for combining different terms into an aggregate, meaning that, whether they disagreed upon whether logical addition should be interpreted as inclusive or exclusive disjunction, they all nevertheless agree that logical addition is a disjunction [Peirce 1883, iii]. Boole and Venn alone among them, he notes, argue in favor or logical addition as exclusive disjunction [Peirce 1883, iv]. Peirce also adds that both he and MacColl also "find it absolutely necessary" to introduce as well a new sign to represent existence. Boole's notation permits assertions that "some description of a thing does *not* exist," but does not allow the possibility of asserting that anything *does* exist [Peirce 1883, iv]. Moreover, the use of the sign of equality, as used by Boole, in expressions such as $xy = 0$ of the E-proposition "No xs are ys" was reformulated in the *Laws of Thought* [Boole 1854, 61–64] as $y = \nu(1 - x)$, asserting with "No xs are ys" and being converted to "All xs are not ys." But, as Augustus De Morgan [De Morgan 1842a, 13], [De Morgan 1842b, 288] pointed out equality is a complex relation, so that the expression "$A = B$" asserts *both* that all As are Bs *and* that all Bs are As. For these reasons, Peirce wrote, he and McColl "make use of signs of inclusion and of non-inclusion" [Peirce 1883, iv]. Peirce writes "Griffin $\overline{\prec}$ Aquatic" to mean that some animals are not aquatic, or that a non-aquatic animal does exist. McColl's notation is not, he tells us, essentially different. At this point, Peirce reiterates a common complaint of logicians of the day, that there is neither a uniform system of notation nor a uniform determination of the most basic propositions and the respective relations or logical connectives holding between them.

The contributions to *Studies in Logic* by Christine Ladd-Franklin [Ladd-Franklin 1883] and Oscar Howard Mitchell (1851-1889) [Mitchell 1883], operating with simplifications of Boole's logic by Schröder and MacColl, helped, in Peirce's estimate to improve the Boolean logic further, and found it "surprising to see with what facility their methods yield solutions of problems more intricate and difficult than any that have hitherto been proposed" [Peirce 1883, v]. Ladd-Franklin [Ladd-Franklin 1883, 23], in considering the copula for her algebra of logic, noted that whereas Jevons considered the two propositions "The sea-serpent is not found in the water" and "The sea-serpent is not found out of the water" to be mutually contradictory, MacColl, Venn, and Peirce were led to consider the existential import of propositions, asserting that a universal proposition can be contradicted only by a particular proposition, and a particular proposition contradicted only by a universal propositions, on the basis of which they concluded that the two propositions "The sea-serpent is not found in the water" and "The sea-serpent is not found out of the water" taken together allowed the inference that sea-serpents do not exist. It is in turn on this basis that Peirce held that, from the formal algebraic perspective, there is no distinction to be made between hypothetical and categorical propositions, and that the Barbara syllogism can be reformulated as $((S \multimap\!\!< M)$ & $(M \multimap\!\!< P)) \multimap\!\!< (S \multimap\!\!< P)$. Another difference between Jevons on the one hand and MacColl (and Peirce) on the other that Ladd-Franklin [Ladd-Franklin 1883, 24] makes is that, while for Jevons, a letter c is a representation of an indefinite class symbol, every letter used by MacColl denotes a distinct proposition, and as MacColl's $a : b$ states that if some object is an a, it also expresses that any statement a implies statement b, as does Peirce's $a \multimap\!\!< b$. The singular advantage that Ladd-Franklin claims for Peirce's "$\multimap\!\!<$" over MacColl's ":" is that the former is, in its graphic design, asymmetrical, and therefore more adept at designating the asymmetrical nature of the relation which it is designed to express. Ladd-Franklin does not address the historical question of priority, or even the broader question of the direction of influence between Peirce and MacColl in "On the algebra of logic," but is chiefly concerned to explore the comparative merits of the offerings of the contemporary algebraic logicans, and to set forth her own contributions in accordance with what she was learning in Peirce's Johns Hopkins courses. (Indeed, the ahistoricality of mathematicians engaged in leading-edge research is, proverbially, nearly universal.)

In her discussion on the methods for elimination of superfluous or redundant terms form logical equations, [12] Ladd-Franklin notes [Ladd-Franklin 1883, 39] that there is a significant difference between the Boolean and classical squares of oppositions, and in particular addresses the question of the existential import of propositions that was a major issue in MacColl's published corpus (as well as an issue in his exchanges on the subject with Bertrand

12. For the historical background and exposition of elemination of terms in formulas of algebraic logic, see [Green 1991].

Russell). As we know, the modern or Boolean square of opposition and the determination of which inferences are valid rests upon the acceptance of empty sets, whereas the traditional (Aristotelian) square and the determination of the validity of inferences presupposes that there are no empty classes involved. In explaining that: "Those syllogisms in which a particular conclusion is drawn from two universal premises become illogical when the universal proposition is taken as not implying the existence of its terms," [Ladd-Franklin 1883, 39], Ladd-Franklin cites both Peirce's "On the algebra of logic" [Peirce 1880] and MacColl's "Symbolical reasoning" [MacColl 1880a]. In this particular instance, the two articles being nearly contemporaneous, no inference whatever is ventured by Ladd-Franklin concerning possible influence of either author on the other. What is patently clear is that Ladd-Franklin regarded Peirce and MacColl as intellectual companions, sharing many of the same concepts and methods in their respective, if presumably independent, development of logic, and both superior to the techniques devised by Jevons. With this evaluation in mind regarding the techniques and their applications to the problems which she considers, Ladd-Franklin adopts those, in her own work, of MacColl, and, finding MacColl's superior to that of Jevons, but more complicated than the alternative that she offers in its place, she writes:

> Complicated problems are solved with far more ease by Mr. McColl than by Mr. Jevons; but that is not because the method of excluded combinations is not, when properly treated, the easiest method. A method of implications, such as that of Mr. McColl, is without doubt more natural than the other when universal premises are given in the affirmative form, but the distinction which it preserves between subject and predicate introduces a rather greater degree of complexity into the rules for working it. [Ladd-Franklin 1883, 51]

Likewise, Mitchell remarks that, apart from the notational difference between Peirce's " $\prec$ " over MacColl's ":" if symbolizing the copula, they both "have given algebraic methods in logic, in which the terms of these propositions are allowed to remain on both sides of the copula" [Mitchell 1883, 96]. Mitchell adds the additional information that:

> Mr. Hugh McColl, in his papers on logic in the "Proceedings of the London Mathematical Society" [Vol. IX, et seq.], has been using a notation for the copula identical in meaning with that of Mr. Peirce. He uses a colon to denote implication, instead of $\prec$. Mr. Peirce has recently told me that Mr. McColl justifies his use of the colon by its mathematical meaning as a sign of division. Thus *Barbara* and *Celarent* are:
>
> $$\begin{array}{ll} m:p & m:\bar{p} \\ s:m & s:m \\ \therefore\ s:p & \therefore\ s:\bar{p}, \end{array}$$
>
> and the analogy to division is obvious. [Mitchell 1883, 101]

Peirce's choice of notation is dictated by the breakdown of the analogy upon which MacColl's selection of notation depends. As Mitchell explains:

> [T]his analogy exists only in the two universal moods of the first figure. Thus *Cesare* and *Festino* are
>
> $$\begin{array}{ll} p : \bar{m} & p : \bar{m} \\ s : m & s \div \bar{m} \\ \therefore s : \bar{p} & \therefore s \div p, \end{array}$$
>
> where $\div$ is the negative copula. And the analogy to division is wanting. [Mitchell 1883, 101–102]

By far the best known of MacColl's work is his advocacy, and development, of nonclassical logic, that is, his three-dimensional, four-valued logic. There are arguments as to whether MacColl's extension of propositional logic should be considered as a modal propositional logic or as a trivalent propositional logic, since it is open to question whether his classifications *true* and *false, certain, impossible* and *variable* were intended to be truth values or modal operators. MacColl's mature system of modal logic was developed in the years 1895-1905 [Astroh & Read 1998, ii] but can be traced back to at least as far back as MacColl's series of papers in *Mind* on "Symbolical reasoning" [MacColl 1880a, 54], in which MacColl presents the denial of $a' + b$ as the equivalent to ab' as a definition of *certainty* or *actuality* (see [Goldblatt 2006, 4]). Fania Cavaliere argued that, working with his operators $\tau, \iota, \varepsilon, \eta, \theta$ to express *true, false, certain, impossible,* and *variable* respectively, and defining "p implies q" as "p and not-q is impossible," MacColl developed a modal propositional logic, thereby anticipating enroute C. I. Lewis's definition of strict implication [Cavaliere 1996]. Peter Simons [Simons 1998] argues, against Nicholas Rescher [Rescher 1969, 4], that MacColl's logic cannot be considered a many-valued logic. It is more properly understood as a modal probability logic.[13] MacColl for his part accounted the failure by Boole and Peirce (and Schröder) to distinguish between an argument (or its contents) and its probability as a serious flaw [Hailperin 1996, 132–134]. But this effectively reduces proofs from distinctions between validity and invalidity to degrees of validity. And while it is indeed crucial to distinguish the contents of an argument and its truth-value, a distinction which is already present in Peirce when he introduced truth functional analysis (beginning in 1884 and perhaps stimulated by Christine Ladd-Franklin [Ladd-Franklin 1883, 62*ff.*] and her study of Jevons [Jevons 1877, 162*ff.*]),[14]

13. In more recent terms, we would be more inclined to suggest that many-valued logics, modal logics, and probability logics all share the same algebraic structure, as *generalized Galois logics* (ggl), in which the main connective is a fusion of implication and some other specialized reflexive, transitive, and asymmetrical relation, with the generalized connective $\circ$ being such that $a \leq b \Leftrightarrow a \circ b$ for any ggl. See [Bimbó & Dunn 2008].

14. [Houser 1991, 20] suggests that Peirce's study of truth-functional logic may have arisen from consideration of dilemmatic reasoning and truth-functional analysis of material implication, in 1884 (MS 527, Part II of "algebra of logic"; published in

by restricting the elements of his logic to propositions, MacColl does not have the capability of interpreting the elements of his system as either terms, sets (or classes), or propositions in the same way that Peirce and Schröder could. Likewise, he does not have the capability to take his basic connective, depending upon context, as Peirce and Schröder were able to do, as either the copula, or class inclusion (or set membership), or implication (or inference).[15]

What is not in dispute is that, as we have already noted, MacColl placed the emphasis in logic upon propositions, and insisted that the basic logical relation is implication between propositions, rather than class inclusion. He investigated the details in his two series of papers "The calculus of equivalent statements" and "Symbolic reasoning," devising a "pure [propositional] logic."

The same questions could perhaps be raised concerning the trivalent logics devised by Peirce, as to whether there is a modal content to the third truth-value. What is evident is that truth-values were paramount, regardless of how those truth values might be modally interpreted. In discussing Russell's criticisms of MacColl's definition of variables, Max Harold Fisch (1900-1995), who devoted much of his professional career to the study of Peirce, and Atwell Rufus Turquette, noted that MacColl's variables are similar to Peirce's concept of possibility [Fisch & Turquette 1966, 80].

For our purposes, however, the primary question is not concerning the interpretation of the classification of propositions or the issue of priority, but rather the issue of the possible influence on Peirce of MacColl's work on non-classical propositions and their logic. Bertrand Russell, as is too well known to record here in any detail, engaged in debates with MacColl on these issues (as well as many others) in the first few years of the twentieth century. It is debatable whether or not Russell countenanced non-classical logics. The general consensus is that he did not. What is probably indisputable is that, whatever consideration (if any), whether positive or negative, that Russell gave to non-classical logics, was stimulated in all probability by his encounters on the matter with MacColl, rather than through the offices of any other source.[16]

Fisch and Turquette wrote that: "It would be interesting to know exactly what circumstances stimulated Peirce's successful solution of the problem of

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[Peirce 1993, 107–109]), along with his rejection of syllogistic. In "On the algebra of logic: A contribution to the philosophy of notation" [Peirce 1885] Peirce used his connective ($-\!\!<$) for material implication and had introduced truth-functional analysis.

15. For example, in [Pierce 1870], Pierce introduced the same sign, "illation" ($-\!\!<$) to represent both implication and class inclusion. He was criticized for this by Russell [Russell 1903, 187]; and Schröder too, following Peirce, used the same symbol, subsumption ($\in$), for both class inclusion and implication, for which he, in turn, was criticized by Frege [Frege 1895].

16. See in particular [Anellis 2009] on the sources from which Russell encountered nonclassical logic. See [Anellis 2009, 153–218], especially § 11: "Peirce again, and MacColl and Bradley" [Anellis 2009,187–193] and § 12: "More on MacColl and Bradley" [Anellis 2009,193–195].

triadic logic" [Fisch & Turquette 1966, 82]. In the absence of concrete extant evidence, Fisch was left to conjecture whether, and if so, when and where, his thinking in terms of the possibility of developing a non-classical logic may have been stimulated by MacColl. Thus, Fisch and Turquette wrote that:

> For example, could it have been that Peirce had been following the controversy in *Mind* between Bertrand Russell and Hugh MacColl, prompted by Russell's (1906) review of MacColl's (1906) *Symbolic Logic* and MacColl's (1907) reply? If so, late in 1908 or early in 1909, Peirce might have seen MacColl's (1908) note entitled " 'If' and 'Imply" ' which had appeared in January 1908. [17] In this note, MacColl considers the difference between his and Russell's treatment of implication. He indicates that for "nearly thirty years" he has been "vainly trying to convince" logicians of the errors involved in equating "implication" with what is now called "Russell's material implication." MacColl then asks the following question:
>
> > Is it too much to hope that this test case will at last open the eyes of logicians to the necessity of accepting my three-fold division of statements $(\varepsilon, \eta, \theta)$ with all its consequences?
>
> [Fisch & Turquette 1966, 82–83]

Fisch and Turquette go on to suggest that it would be speculative to attempt to determine the direction of influences between Peirce and MacColl with respect at least to Peirce's triadic logic and MacColl's three-dimensional logic [Fisch & Turquette 1966, 83]. They can confidently assert that they were interested in one another's work for a considerable period of time, and they quote a letter to Peirce dated May 16, 1883 in which MacColl wrote that: "It will be a great pleasure indeed to me if you can stay a little while in Boulogne on your way to England. It is not often that I have the opportunity of making the personal acquaintance of my correspondents in logic and mathematics" [MacColl 1883]. Much later, in a draft of a letter dated November 16, 1906, Peirce writes to MacColl that:

> Although my studies in symbolic logic have differed from yours in that my aim has not been to apply the system to the working out of problems, as yours has, but to aid in the study of logic itself, nevertheless I have always thought that you alone, so far as I know, except myself, have understood how the matter ought to be treated by making the *elements* propositions or predicates and not common nouns. [Peirce 1906]

Peirce concludes by noting that he has learned of a new book just published on symbolic logic which was reviewed in the November first issue of *Nature* [Anonymous 1906b], and hints that he should like to receive a copy. This

17. Fisch and Turquette are clearly referring here to the first of two parts of MacColl's [MacColl 1908].

letter is also cited by Michael Astroh [Astroh 1998, 170] as evidence that Peirce was in agreement with MacColl that the traditional account by logicians of subject and predicate in terms of general nouns was erroneous when reflected in logical form.[18]

The book to which Peirce refers, is, of course, MacColl's *Symbolic Logic* [MacColl 1906]. The account of MacColl's book begins: "WHETHER Mr. MacColl is the Athanasius of symbolic logic or only its Ishmael, the fact remains that he seems unable to come to an agreement with other exponents of the subject. But he contends that his system in the elastic adaptability of its notation bears very much the same relation to other systems (including the ordinary formal logic of our text-books) as algebra bears to arithmetic" [Anonymous 1906b, 1].[19] After noting that its contents had been previously published in journals, the reviewer continues that points on which MacColl lays

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18. In light of the claims by Francis Herbert Bradley (1846-1924), referring to his *Principles of Logic* [Bradley 1883], as expressed in his letter to Bertrand Russell of 27 September 1914, that "I always have believed that in 1883 in the Logic I pointed out a number of inferences which fell outside the category of subject and attribute and pointed out again that there was nevertheless a form in every possible inference," quoted in [Keen 1971, 10]. As Francisco A. Rodríguez-Consuegra made patently clear, Bradley, as much as Russell, was "an enemy of this [subject-predicate] mode of analysis," and he explains that, for Bradley, the subject-predicate analysis "was the basis for traditional logic, and therefore to be rejected," and that his writings "contain many examples of this rejection," because, judgments as relational, the only distinctions that apply between subject and predicate are grammatical [Rodríguez-Consuegra 1992, 95]. See also [Chalmers & Griffin 1997].

While a discussion of interactions of MacColl and Bradley, MacColl and Russell, and Bradley and Russell are quite tangential to the discussion of MacColl's influences on Peirce and Schröder, I suggest that it might nevertheless be useful to also examine what influence MacColl's work in logic may have had on Bradley, and to investigate Bradley's studies of MacColl's work. Exchanges of correspondence between Bradley and MacColl, and between Russell and Bradley have been published in [Keene 1999].

Regardless of the issue of the connections between Bradley and MacColl, these discussions between Peirce and MacColl, and Bradley and Russell, ought, when taken collectively, dispell the claims, perpetuated by much of twentieth-century historiography of logic, that it was Frege who, single-handedly, broke the Aristotelian tradition of treating logical propositions exclusively in subject-predicate form. This does not, of course, contradict the role or significance of Frege in replacing the subject-predicate syntax with the function-theoretic syntax of mathematical logic.

An anonymous reviewer of MacColl's *Symbolic Logic* in the *Educational Times* [Anonymous 1906a] remarked on the significance of MacColl's opposition to the subject-predicate form in logical syntax. It would be interesting, if perhaps also fruitless, to speculate whether this anonymous author might actually have been Peirce, or Bradley, or Russell.

19. MacColl's book was considered in that review along with *The Development of Symbolic Logic* [Shearman 1906] of Arthur Thomas Shearman (1866-1937); *An Introduction to Logic* [Joseph 1906] of Horace William Brindley Joseph (1867-1943); and *Thought and Things, or Genetic Logic* [Baldwin 1906] of James Mark Baldwin (1861-1934).

"considerable stress, and in which he does not command the uniform ascent of the other symbolic logicians," are:

> (*a*) that he takes statements and not terms to be in all cases and necessarily the ultimate constituents of symbolic reasoning; (*b*) that he goes quite beyond the ordinary notation of the symbolists in classifying propositions according to such attributes as true, false, certain, impossible, variable; (*c*) that in regard to the existential import of propositions, while other symbolists define the null class o as containing no members, and understand it as contained in every class, real or unreal, he, on the other hand, defines it as consisting of the null or unreal members o_1, o_2, o_3, &c., and considers it to be extended over every real class... [Anonymous 1906b, 1]

The anonymous reviewer for *Nature* does not name the "other symbolic logicians" from whom MacColl differs on these three issues, although, since a criticism by Shearman is mentioned by the reviewer in the context of the discussion of Shearman's *Development of Symbolic Logic*, we can be assured that he, at least, is one of the dissenters had in mind by the reviewer, it being noted that "the doctrines of Prof. Jevons and Mr. MacColl are subjected" by Shearman "to some severe criticisms..." [Anonymous 1906b, 1]. It is noted in particular that Shearman "rejects all attempts to deal with any but assertoric propositions, and holds that if Mr. MacColl wishes to work with such data as probable and variable he should introduce new terms" [Anonymous 1906b, 1].

Fisch and Turquette conclude, with respect to their work on nonclassical logic, that: "As far as it is now known, however, there is not sufficient concrete evidence available to make possible an accurate account of the exact relationship between MacColl's work in three-dimensional logic and Peirce's investigations of triadic logic. The solution to this problem will have to await future historical research" [Fisch & Turquette 1966, 83]. We are obliged then to turn, on this matter, to the findings of Rahman and Redmond [Rahman & Redmond 2008] and others to glean what we can, or to hope that new extant evidence will appear.

The question of MacColl's influences on Peirce leading to Peirce's consideration of the possibility of devising a nonclassical logic, and then to his efforts to do so, are indirect at best. We look in vain for an instance in which Peirce states that he was led in this direction by reading MacColl's work. And, as questions of influence can be shadowy, the best we can do is examine the extant written evidence.

We know that Peirce had already considered paradoxes of self-reference in "Grounds of the validity of the laws of logic" [Peirce 1869], in particular dealing with the ramifications of the proposition "This proposition is false," and offering an analysis closely akin to that of Paul of Venice (Paulus Venetus; Paolo di Venezia; 1369-1429), but giving a proof of assumptions that Paul merely assumed, and demonstrating that the analysis of the underlying premise is inadequate, namely that it states merely that the proposition is false, whereas, in reality, the underlying premise also includes the presupposition that the statement is true. Here, perhaps, we can see the roots of the trivalent logic that Peirce eventually devised beginning in the 1890s.

We also know that Peirce also raised the question of the possibility of developing non-classical logics as early as the 1890s. He wrote that by 1895 he had entertained the idea of toying with, and altering, the laws of logic and even of doing without some of them, as he suggested in a letter to Judge Francis Calvin Russell (1838-*ca.*1920) [no relation to Bertrand Russell], quoted by Paul Calvin Carus (1852-1919) in [Carus 1910a, 44–45]. In that letter, Peirce wrote that "before I took up the general study of relatives, I made some investigation into the consequences of supposing the laws of logic to be different from what they are." He did not consistently, persistently or thoroughly follow through, although he left manuscript notes in which his "tinkering" appears. On the other hand, Peirce left no doubt, as Carus assured his readers in "Non-Aristotelian logic" [Carus 1910b], that he did *not,* as some readers of the original Carus paper [Carus 1910a, 44–45] might have been led to surmise, consider the possibility of non-Aristotelian logics to be in any wise "lunatic." To the contrary, Peirce indicated that had he deigned to pursue the issue, he might have gained some insight into features of logic that might be overlooked (Peirce quoted at [Carus 1910b, 158]). Even earlier, George Boole, in his *Laws of Thought,* after proving on the previous page that $x^2 = x$, showed that $x^2 = x$ is impossible, and added, in the attached footnote, that, had it been accepted, it could have opened the possibility of a non-bivalent logic in which "the law of thought might have been be different from what it is" [Boole 1854, 50]. Peirce was certainly familiar with Boole's *Laws of Thought,* and could have been thinking of Boole's footnote. His own comment (as quoted by Carus [Carus 1910a, 45]) of entertaining the consequences of "supposing the laws of logic to be different from what they are," is clearly an echo of Boole's similar remark. This leaves open the possibility that Boole, together with MacColl, or even possibly Boole alone, could have led Peirce to take seriously the possibility for constructing non-classical logics.

Consider as examples of Peirce's "tinkerings" with the laws of logic the fragments of Peirce's manuscript *Logic Notebook* (1865-1909), in particular those dating from at least early 1909 if not somewhat earlier (MS 339:440–441, 443; see [Peirce 1849-1914]), which had been examined by Fisch and Turquette On the basis of this work. Fisch and Turquette [Fisch & Turquette 1966, 72] concluded that by 23 February 1909 Peirce was able to extend his truth-theoretic matrices to three-valued logic, thereby anticipating both Jan Łukasiewicz (1878-1956), in "O logice trójwartosciowej" [Łukasiewicz 1921], and Emil Leon Post (1897-1954), in "Introduction to a general theory of elementary propositions" [Post 1921], by a decade in developing the truth table device for triadic logic and multiple-valued logics respectively.

Peirce's influences on MacColl

The influence which Peirce and Schröder had upon MacColl, on the other hand, was comparatively tenuous. In particular, as Ivor Grattan-Guinness [Grattan-Guinness 1998, 11] noted, MacColl used Peirce's symbol for implication in a passage of his book, where he also noted Schröder's [MacColl 1906, 78–80], but he did not discuss or use their systems. We may attribute this at least in part to MacColl's comparative isolation from the mainstream of the logical research community and from his unfamiliarity with many of the languages in which some of the work of active researchers was being carried out. Thus, for example, Grattan-Guinness [Grattan-Guinness 1998, 10] noted that lack of Italian left MacColl unable to read the work of Peano and his school, while his lack of German rendered him unable to take advantage of the work of Schröder, although MacColl attended and delivered a talk [MacColl 1901] at the same International Congress of Philosophy in Paris in 1900 at which Schröder also spoke [Schröder 1901].[20] Thus, in a letter to Russell of 28 June 1901 (quoted in [Russell 1993, 239–240]), MacColl professes his ignorance of the logic of relations and, alluding to the first two volumes of the *Vorlesungen über die Algebra der Logik* [Schröder 1890-1905], his inability to read the work of Schröder because he cannot read German. The congress was MacColl's best, and perhaps last, opportunity to get his work before an influential group of mathematical logicians, and his talk, "La logique symbolique et ses applications," provided them with a survey of his logical system. MacColl made no reference in his own work to the work of Schröder, nor to Schröder's congress paper.

The influences which Peirce and Schröder had on MacColl were, at best, minimal. He was aware of their work, at one point using Peirce's notation for implication in his own *Symbolic Logic* [MacColl 1906, 78–80] and taking note of the existence of Schröder's work, but failing even at that juncture to provide his own exposition, or even summary, of that work. MacColl indeed went to great lengths to differentiate his work from that of Peirce and Schröder. The entirety of his "A note on Prof. C. S. Peirce's probability notation of 1867" [MacColl 1881] is devoted to establishing that any similarity between his own notation for implication and Peirce's illation is purely coincidental; thus, the Secretaries of the London Mathematical Society assert, in MacColl's name, that MacColl, referring to his article "The calculus of equivalent statements (fourth paper)" [MacColl 1880b], has shown that:

> the apparent coincidence of notation, in some few particulars, between himself and Prof. Peirce, was entirely accidental, and that

20. [Lovett 1900-1901] is a summary of the the talks on mathematics presented at the Congress. MacColl's and Schröder's talks and discussions are summarized at [Lovett 1900-1901, 166–168], with Couturat raising questions for MacColl regarding the differences between Boole's treatment of probability and MacColl's [Lovett 1900-1901, 176–177]. If there were any discussions between MacColl and Schröder, or even whether either attended the other's talk, is not recorded in [Lovett 1900-1901].

> Mr. McColl was not at that time acquainted with Prof. Peirce's paper. In fact, the revised fourth paper ("On Probability Notation," *Proceedings of the London Mathematical Society,* Vol. xi., n° 163) was communicated to the Society about nine months before the Author read Prof. Peirce's paper. [MacColl 1881]

The assumption that we may infer is that, whereas Peirce and Schröder were willing to acknowledge debts to MacColl, this willingness was not, evidently, reciprocated. Alternatively, we might just as easily take MacColl at face value and allow the possibility that he was largely uninfluenced by either Peirce or Schröder, or, at the very least that, if he was influenced by either or both, he failed to recognize it because his conception of logical systems and their underlying structures were significantly different from theirs, namely, that whereas he considered implication to be fundamental to logical form and the basic syntactic elements of logic were propositions, Peirce and Schröder considered illation and subsumption respectively, understood as generalized reflexive, transitive, assymmetrical relations, to be fundamental to the algebraic structure of logical systems and the basic syntactic elements to be the relata, i.e. the elements of the relations, interpretable as either terms, sets, classes, or propositions, depending upon the particular requirements of the system in its application.

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References

ANELLIS, IRVING H.

2009 Russell and his sources for non-classical logics, *Logica Universalis,* 3, 153–218.

ANONYMOUS

1906a Review of Hugh MacColl's *Symbolic Logic and its Applications* (London: Longmans, Green, & Co, 1906), *The Educational Times and Journal of the College of Preceptors,* 59, June, 261.

1906b Some recent works on logic: Review of (1) Hugh MacColl's *Symbolic Logic and its Applications* (London: Longmans, Green, & Co, 1906); (2) A.T. Shearman's *The Development of Symbolic Logic* (London: Williams and Norgate, 1906); (3) H.W.B. Joseph's *An Introduction to Logic* (Oxford, Clarendon Press, 1906); (4) James Mark Baldwin's

Thought and Things, or Genetic Logic (London: Swan Sonnenschein and Co., 1906), *Nature,* 75 (1931), 1 November, 1–2.

ASTROH, MICHAEL

1998 MacColl's evolutionary design of language, *Nordic Journal of Philosophical Logic,* 3, 141–173.

ASTROH, MICHAEL, & READ, STEPHEN (EDS.)

1998 *Hugh MacColl and the Tradition of Logic,* special issue of the *Nordic Journal of Philosophical Logic,* 3 (1), December.

BAIN, ALEXANDER

1870 *Logic,* 2 vols., London: Longmans, Green, Reader, & Dyer.

BALDWIN, JAMES MARK

1906 *Thought and Things: A Study of the Development and Meaning of Thought; or Genetic Logic,* I, *Functional Logic, or Genetic Theory of Knowledge,* New York: Macmillan.

BIMBÓ, KATALIN, & DUNN, JON MICHAEL

2008 *Generalized Galois Logics: Relational Semantics of Nonclassical Logical Calculi,* Stanford: CSLI Publications.

BOOLE, GEORGE

1854 *An Investigation of the Laws of Thought, on which are founded the Mathematical Theories of Logic and Probabilities,* London: Walton & Maberly (reprinted with an introduction by M. Friend, New York: Barnes & Noble, 2006).

BOS, HENK J. M.

1974 Differentials, higher-order differentials and the derivative in the Leibnizian calculus, *Archive for History of Exact Sciences,* 14, 1–90.

BRADLEY, FRANCIS HERBERT.

1883 *The Principles of Logic,* London: Kegan Paul, Trench.

CARUS, PAUL C.

1910a The nature of logical and mathematical thought, *The Monist,* 20, 33–75.

1910b Non-Aristotelian logic, *The Monist,* 20, 158–159.

CAVALIERE, FANIA

1996 L'opera di Hugh MacColl alle origine delle logiche non-classiche, *Modern Logic,* 6, 373–402.

Chalmers, Mélanie & Griffin, Nicholas

1997 Russell's marginalia in his copy of Bradley's *Principles of Logic*, *Russell*, 17, 43–70.

Couturat, Louis

1899 La logique mathématique de M. Peano, *Revue de métaphysique et de morale*, 7, 616–646.

Delboeuf, Joseph-Rémi Léopold

1876 Logique algorithmique : exposé de la logique déductive au moyen d'un système des signes appliqués à la logique, *Revue philosophique de la France et de l'étranger*, 1, 225–252, 335–355, 545–595 (reprinted as : *Logique algorithmique*, Liège: Librairie de J. Desoer – Bruxelles: Librairie C. Muquardt, 1877).

De Morgan, Augustus

1842a *The Differential and Integral Calculus: containing differentiation, integration, development, series, differential equations, differences, summation, equations of differences, calculus of variations, definite integrals, with applications to algebra, plane geometry, solid geometry, and mechanics: also, elementary illustrations of the differential and integral calculus*, London: Baldwin & Cradock.

1842b On the foundation of Algebra, *Transactions of the Cambridge Philosophical Society*, 7, 173–187, 287–300.

Fisch, Max Harold & Turquette, Atwell R.

1966 Peirce's triadic logic, *Transactions of the Charles S. Peirce Society*, 2, 71–85 (reprinted *in* M. H. Fisch, *Peirce, Semeiotic, and Pragmatism: Essays*, edited by K. L. Ketner & C. J. W. Kloesel, Bloomington: Indiana University Press, 1986, 171–183).

Frege, Gottlob

1895 Kritische Beleuchtung einiger Punkte in E. Schröders *Vorlesungen über die Algebra der Logik*, *Archiv für systematische Philosophie*, 1, 433–456.

Goldblatt, Robert

2006 Mathematical modal logic: a view of its evolution, *in* D. M. Gabbay & J. Woods (eds.), *Logic and Modalities in the Twentieth Century*, vol. 5 of: *Handbook of the History of Logic*, Amsterdam: North-Holland (Elsevier), 1–98.

Grassmann, Robert

1872a *Die Formenlehre oder Mathematik in strenger Formulentwicklung*, Stettin: R. Grassmann.

1872b *Die Begriffslehre oder Logik: Zweites Buch der Formenlehre oder Logik,* Stettin: R. Grassmann.

GRATTAN-GUINNESS, IVOR

1998 Are other logics possible? MacColl's logic and some English reactions, 1905-1912, *Nordic Journal of Philosophical Logic,* 3, 1–16.

GREEN, JUDY

1991 The problem of elimination in the algebra of logic, *in* T. Drucker (ed.), *Perspectives on the History of Mathematical Logic,* Boston – Basel – Berlin: Birkhäuser Verlag, 1–9.

HAILPERIN, THEODORE

1996 *Sentential Probability Logic,* Bethlehem/London: Lehigh University Press/Associated Universities Press.

HOUSER, NATHAN

1987 Peirce's early Work on the algebra of logic: remarks on Zeman's account, *Transactions of the Charles S. Peirce Society,* 23, 425–440.

1989 Introduction, *in* C. J. W. Kloesel (ed.), *Writings of C. S. Peirce: A Chronological Edition,* 4: 1879-1884, Bloomington – Indianapolis: Indiana University Press, xix–lxx.

1991 Peirce and the law of distribution, *in* T. Drucker (ed.), *Perspectives on the History of Mathematical Logic,* Boston – Basel – Berlin: Birkhäuser, 10–32.

JEVONS, WILLIAM STANLEY

1864 *Pure Logic or the Logic of Quality apart from Quantity,* London: Edward Stanford.

1877 *The Principles of Science, a Treatise on Logic and Scientific Method,* 2nd ed., London: Macmillan (3rd ed., 1879).

1881 Recent mathematico-logical Memoirs, *Nature,* 23, 485–487.

JOSEPH, HORACE WILLIAM BRINDLEY

1906 *An Introduction to Logic,* Oxford: Clarendon Press.

KEEN, C. N.

1971 The interaction of Russell and Bradley, *Russell,* 3, Autumn, 7–11.

KEENE, CAROL A. (ED.)

1999 *F. H. Bradley: Selected Correspondence. June 1872-December 1904,* Bristol: Thoemmes.

LADD-FRANKLIN, CHRISTINE

1883 On the algebra of logic, *in* C. S. Peirce (ed.), *Studies in Logic,* Boston: Little, Brown & Company, 17–71.

1889 On some characteristics of symbolic logic, *The American Journal of Psychology,* 2, 543–567.

LEWIS, CLARENCE IRVING

1912 Implication and the algebra of logic, *Mind,* 21, 522–531.

1913 A new algebra of implications and some consequences, *The Journal of Philosophy, Psychology and Scientific Methods,* 10, 428–438.

1918 *A Survey of Symbolic Logic,* Los Angeles: University of California Press.

1960 *A Survey of Symbolic Logic,* New York: Dover Publications (abridged version of the 1918 original edition)

LIARD, LOUIS

1877a Un nouveau système de logique formelle : M. Stanley Jevons, *Revue philosophique de la France et de l'étranger,* 3, 277–293.

1877b La logique algèbrique de Boole, *Revue philosophique de la France et de l'étranger,* 4, 285–317.

1878 *Les Logiciens anglais contemporains,* Paris: Ballière et C[ie] (5[e] ed., Paris: Félix Alcan, 1907).

LOVETT, EDGAR ODELL

1900-1901 Mathematics at the International congress of philosophy, Paris, 1900, *Bulletin of the American Mathematical Society,* 7, 157–183.

ŁUKASIEWICZ, JAN

1921 O logice trójwartosciowej, *Ruch Filozoficzny,* 5, 169–170.

MACCOLL, HUGH

1877-1878a The calculus of equivalent statements and integration limits, *Proceedings of the London Mathematical Society,* 9, 9–20.

1877-1878b The calculus of equivalent statements (2[nd] paper), *Proceedings of the London Mathematical Society,* 9, 177–186.

1878-1879 The calculus of equivalent statements (3[rd] paper), *Proceedings of the London Mathematical Society,* 10, 16–28.

1880a Symbolical reasoning, *Mind,* n. s., 5, 45–60.

1880b The calculus of equivalent statements (fourth paper), *Proceedings of the London Mathematical Society,* 11, 112–121.

1881 A note on Prof. C. S. Peirce's probability notation of 1867, *Proceedings of the London Mathematical Society,* 12, 102.

1883 Letter to C. S. Peirce, 16 May, *Peirce Archives,* Widener Library, Harvard University, USA (Robin catalog #MS L 261).

1896 The calculus of equivalent statements (5[th] paper), *Proceedings of the London Mathematical Society,* 28, 156–183.

1901 La logique symbolique et ses applications, *Bibliothèque du I^er Congrès international de philosophie,* Paris: A. Colin, 135–183.

1902a Logique tabulaire, *Revue de métaphysique et de morale,* 10, 213–217.

1902b Symbolic reasoning (IV), *Mind,* 11, 352–368.

1905 Symbolic reasoning (VI), *Mind,* 14, 74–81.

1906 *Symbolic Logic and Its Applications,* London: Longmans, Green, & Co.

1907 Symbolic logic (a reply), *Mind,* 16, 470–473.

1908 'If' and 'Imply', *Mind,* n. s., 17, 151–152; 453–455.

1910 Linguistic misunderstandings, *Mind,* 19, 186–199, 337–355.

MITCHELL, OSCAR HOWARD

1883 On a new algebra of logic, *in* C. S. Peirce (ed.), *Studies in Logic,* Boston: Little, Brown & Company, 72–106.

PEANO, GIUSEPPE

1889 *Arithmetices Principia, Nova Methodo Exposita,* Torino: Bocca.

1901 *Formulaire de mathématique,* III, Torino: Bocca.

PECKHAUS, VOLKER

1998 Hugh MacColl and the German algebra of logic, *Nordic Journal of Philosophical Logic,* 3, 17–34.

2004 Schröder's logic, *in* D. M. Gabbay & J. Woods (eds.), The Rise of Modern Logic from Leibniz to Frege, vol. 3 of *Handbook of the History of Logic,* Amsterdam: North-Holland (Elsevier), 557–609.

PEIRCE, CHARLES S.

1849-1914 *The C. S. Peirce Papers,* Manuscript collection in the Houghton Library, Harvard University, USA (Photocopies at the Peirce Edition Project, Indiana University – Purdue University at Indianapolis, USA).

1869 Grounds of the validity of the laws of logic, *Journal of Speculative Philosophy,* 2, 193–208.

1870 Description of a notation for the logic of relatives, resulting from an amplification of the conceptions of Boole's calculus of logic, *Memoirs of the American Academy,* 9, 317–378 (reprinted in: C. J. W. Kloesel (ed.), *Writings of C. S. Peirce: A Chronological Edition,* 4, Bloomington – Indianapolis: Indiana University Press, 1989, 359–429).

1880 On the algebra of logic, *American Journal of Mathematics,* 3, 15–57 (reprinted in: C. J. W. Kloesel (ed.), *Writings of CS. Peirce: A Chronological Edition,* 4, Bloomington – Indianapolis: Indiana University Press, 1989, 163–209).

1885 On the algebra of logic: a contribution to the philosophy of notation, *American Journal of Mathematics,* 7, 180–202.

1906 Letter to Hugh MacColl, November 16; *Peirce Archives,* Widener Library, Harvard University, USA (Robin catalog #MS L 261).

1982 *Writings of C. S. Peirce: A Chronological Edition,* 1: 1857-1866, edited by M. H. Fisch, Bloomington: Indiana University Press.

1993 *Writings of C. S. Peirce: A Chronological Edition,* 5: 1884-1886, edited by C. J. W. Kloesel, Bloomington – Indianapolis: Indiana University Press.

2000 *Writings of C. S. Peirce: A Chronological Edition,* 6: 1886-1890, edited by N. Houser, Bloomington – Indianapolis: Indiana University Press.

2010 *Writings of C. S. Peirce: A Chronological Edition,* 8: 1890-1892, edited by N. Houser, Bloomington – Indianapolis: Indiana University Press.

PEIRCE, CHARLES S. (ED.)

1883 *Studies in Logic,* Boston: Little, Brown & Company (reprinted with an introduction by M. H. Fisch and a preface by A. Eschbach, Amsterdam: John Benjamins Publishing Co., 1983).

POST, EMIL LEON

1921 Introduction to a general theory of elementary propositions, *American Journal of Mathematics,* 43, 163–185.

RAHMAN, SHAHID

2000 Hugh MacColl and George Boole on hypotheticals, *in* J. Gasser (ed.), *A Boole Anthology: Recent and Classical Studies in the Logic of George Boole,* Dordrecht: Kluwer Academic Publishers, 287–310.

RAHMAN, SHAHID, & REDMOND, JUAN

2008 Hugh MacColl and the birth of logical pluralism, *in* D. M. Gabbay & J. Woods (eds.), British Logic in the Nineteenth Century, vol. 4 of: *Handbook of the History of Logic,* Amsterdam: North-Holland (Elsevier), 533–604.

READ, STEPHEN

1998 Hugh MacColl and the algebra of strict implication, *Nordic Journal of Philosophical Logic,* 3, 59–83.

RESCHER, NICHOLAS

1969 *Many-Valued Logic,* New York: McGraw-Hill.

RODRÍGUEZ-CONSUEGRA, FRANCISCO A.

1992 Review of Elizabeth Ramsden Eames's *Bertrand Russell's Dialogue with his Contemporaries* (Carbondale and Edwardsville: Southern Illinois U. P., 1989), Russell, n. s., 12 (1), 93–104.

RUSSELL, BERTRAND

1901 Sur la logique des relations avec des applications à la théorie des séries, *Revue de mathématiques/Rivista di Matematiche,* 7, 115–148.

1903 *Principles of Mathematics,* London: Cambridge University Press (2nd ed., London: Allen & Unwin, 1937; 2nd American ed., New York: W.W. Norton, 1938).

1906 Review of Hugh MacColl's *Symbolic Logic and its Applications* (London: Longmans, Green, & Co, 1906), *Mind,* n. s., 15, 255–260.

1993 Towards the "Principles of Mathematics", 1900-1902, vol. 3 of *The Collected Papers of Bertrand Russell,* edited by G. H. Moore, London – New York: Routledge.

SCHRÖDER, ERNST

1877 *Der Operationskreis des Logikkalkuls,* Leipzig: Teubner (reprinted at: Darmstadt, Wissenschaftliche Buchgesellschaft, 1966).

1890-1905 *Vorlesungen über die Algebra der Logik (Exakte Logik),* 3 vols., Leipzig: B. G. Teubner (reprinted at: New York: Chelsea Publishing Co., 1966).

1901 Sur une extension de l'idée d'ordre, *Bibliothèque du Ier Congrès international de philosophie,* Paris: A. Colin, 235–240.

SHEARMAN, ARTHUR THOMAS

1906 *The Development of Symbolic Logic: A Critical-historical Study of the Logical Calculus,* London: Williams and Norgate.

SIMONS, PETER

1998 MacColl and many-valued Logic, *Nordic Journal of Philosophical Logic,* 3, 85–90.

STYAZHKIN, NIKOLAI IVANOVICH

1969 *History of Mathematical Logic from Leibniz to Peano,* Cambridge, MA. – London: MIT Press.

TILES, MARY

1989 *The Philosophy of Set Theory: An Historical Introduction to Cantor's Paradise,* Oxford-Cambridge, MA: Blackwell (reprinted at: Mineola, N.Y.: Dover Publications, 2004).

VENN, JOHN

1881 *Symbolic Logic,* 1st ed., London: Macmillan and Co. (2nd ed., 1894; reprinted at: Bronx, N.Y: Chelsea Publishing Co., 1971).

WIENER, NORBERT

1916 Mr. Lewis and Implication, *The Journal of Philosophy, Psychology and Scientific Methods,* 13, 656–662.