Microeconomics I
Solutions to Problem Set 4

1.
a) $MP_1 = 4x_1^3x_2^4$; $MP_2 = 4x_1^4x_2^3$; $TRS = \frac{x_2}{x_1}$.
Regarding returns to scale,

$$f(\lambda x_1, \lambda x_2) = (\lambda x_1)^4 (\lambda x_2)^4 = \lambda^4 x_1^4 x_2^4 = \lambda^4 f(x_1,x_2) > \lambda f(x_1,x_2) \text{ for } \lambda > 1$$

thus, increasing returns to scale.
b) $MP_1 = \frac{1}{4}x_1^{\frac{-3}{4}}x_2^{\frac{1}{4}}$; $MP_2 = \frac{1}{4}x_1^{\frac{1}{4}}x_2^{\frac{-3}{4}}$; $TRS = \frac{x_2}{x_1}$.
Regarding returns to scale,

$$f(\lambda x_1, \lambda x_2) = (\lambda x_1)^{\frac{1}{4}} (\lambda x_2)^{\frac{1}{4}} = \lambda^{\frac{1}{2}} x_1^{\frac{1}{4}} x_2^{\frac{1}{4}} = \lambda^{\frac{1}{2}} f(x_1,x_2) < \lambda f(x_1,x_2) \text{ for } \lambda > 1$$

thus, decreasing returns to scale.
c) $MP_1 = \frac{1}{x_1}$; $MP_2 = \frac{1}{x_2}$; $TRS = \frac{x_2}{x_1}$.
Skip the discussion on returns to scale for this case.
d) $MP_1 = 5$; $MP_2 = 3$; $TRS = \frac{5}{3}$.
Regarding returns to scale,

$$f(\lambda x_1, \lambda x_2) = 5\lambda x_1 + 3\lambda x_2 = \lambda (5x_1 + 3x_2) = \lambda f(x_1,x_2) \text{ for } \lambda > 1$$

thus, constant returns to scale.
e) $MP_1 = 0$ if $x_1 > x_2$, 1 if $x_1 < x_2$; $MP_2 = 1$ if $x_1 > x_2$, 0 if $x_1 < x_2$; $TRS = 0$ if $x_1 > x_2$, ∞ if $x_1 < x_2$.
Regarding returns to scale,

$$f(\lambda x_1, \lambda x_2) = \min\{\lambda x_1, \lambda x_2\} = \lambda \min\{x_1, x_2\} = \lambda f(x_1,x_2) \text{ for } \lambda > 1$$

thus, constant returns to scale.

2. If $f(x_1,x_2) = (x_1^b + x_2^b)^c$, with $b, c > 0$, in order for the firm to exhibit increasing returns to scale, $f(\lambda x_1, \lambda x_2) > \lambda f(x_1,x_2)$, for $\lambda > 1$. Thus,

$$f(\lambda x_1, \lambda x_2) = (\lambda^b x_1^b + \lambda^b x_2^b)^c = \lambda^{bc}(x_1^b + x_2^b)^c = \lambda^{bc} f(x_1,x_2)$$

therefore, in order for $f(\lambda x_1, \lambda x_2) > \lambda f(x_1,x_2)$, we need $\lambda^{bc} > \lambda \Rightarrow bc > 1$.

3.
First,

$$f(\lambda x_1, \lambda x_2) = min\{2\lambda x_1 + \lambda x_2, \lambda x_1 + 2\lambda x_2\} = \lambda min\{2x_1 + x_2, x_1 + 2x_2\} = \lambda f(x_1,x_2)$$

Similarly,

$$g(\lambda x_1, \lambda x_2) = \lambda x_1 + min\{\lambda x_1, \lambda x_2\} = \lambda (x_1 + min\{x_1, x_2\}) = \lambda g(x_1,x_2)$$

Finally,

$$f(\lambda x_1, \lambda x_2) = 3(\lambda x_1)^{0.4} + 6(\lambda x_2)^{0.7} = \lambda\left(3\frac{x_1^{0.4}}{\lambda^{0.6}} + 6\frac{x_2^{0.7}}{\lambda^{0.3}}\right) < \lambda f(x_1, x_2)$$

4.

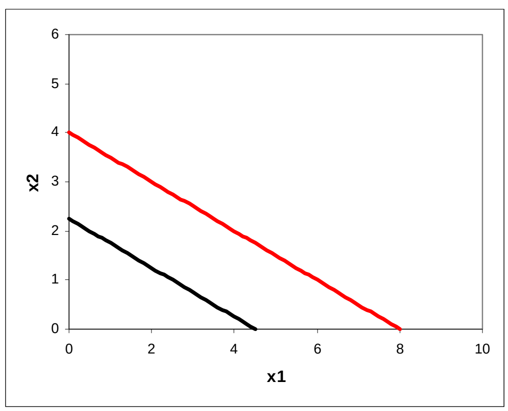
Here $MP_F = 1 - \frac{N}{200}$. In order to maximize profits, it has to be the case that $pMP_F = p_F$, where p_F is the price of fertilizer, and p is the output price. Therefore,

$$4\left(1 - \frac{N}{200}\right) = 1.2 \Rightarrow N = 140$$

5.

Here inputs are perfect substitutes, which implies that we should watch for corner solutions. Furthermore, the production function exhibits decreasing returns to scale.

(a) The expressions for the required isoquants are $9 = 2x_1 + 4x_2$ and $16 = 2x_1 + 4x_2$, which are straight lines:



(b) In the case of an interior solution (the firm using positive amounts of both inputs) it is true that, for every input, $p \cdot PM_i = w_i$. However, given marginal products and input prices, the firm will only use input 2. This is because:

$$\frac{MP_1}{w_1} < \frac{MP_2}{w_2}$$

for any value of x_1, x_2. Hence, the optimal amount of input 2 used comes from $p \cdot PM_2 = w_2$. Thus,

$$4\frac{2}{\sqrt{4x_2}} = 3 \Rightarrow x_2 = \frac{16}{9}$$

and output is $f\left(0, \frac{16}{9}\right) = \sqrt{4 \cdot \frac{16}{9}} = \frac{8}{3}$.

6.
$f(K,L) = \frac{L}{2} + \sqrt{K}$
(a) Regarding returns to scale,

$$f(\lambda K, \lambda L) = \frac{\lambda L}{2} + \sqrt{\lambda K} = \lambda \left[\frac{L}{2} + \sqrt{\frac{K}{\lambda}} \right] < \lambda \left[\frac{L}{2} + \sqrt{K} \right] = \lambda f(K,L)$$

On the other hand,

$$MP_L = \frac{1}{2}$$

(b) If capital is fixed at 4 units and labor is variable, the short-run production function is $f(4,L) = 2 + \frac{L}{2}$. Thus, $MP_L = \frac{1}{2}$ and $APL = \frac{1}{2} + \frac{2}{L}$.

(c) To see what the short-run maximizing profit amount of labor if $w_L = 1$ and the output price is 1 euro per unit notice that if we just apply the condition for an interior profit-maximizing output level, we obtain:

$$p \cdot MP_L = w_L \Rightarrow \frac{1}{2} = 1$$

which is impossible. Hence, $L = 0$. This is because the marginal revenue product of labor is lower than the cost of one unit of labor. In a sense, labor is too expensive a factor. Thus, the firm is best off not using any labor at all.

7. The expression for the short-run production function is $f(16,L) = \sqrt{\min\{16, L\}}$. This implies that

$$f(16,L) = \begin{cases} \sqrt{L} \text{ if } L \leq 16 \\ 4 \text{ if } L > 16 \end{cases}$$

(a) Given the short-run production function, the expression for the short-run marginal product of labor is:

$$MP_L = \begin{cases} \frac{1}{2\sqrt{L}} \text{ if } L \leq 16 \\ 0 \text{ if } L > 16 \end{cases}$$

(b) In the short run, given p and w, the firm chooses L to maximize profits:

$$\max_{L \geq 0} p f(16,L) - wL - 16r$$
$$\text{s.t. } f(16,L) = \begin{cases} \sqrt{L} \text{ if } L \leq 16 \\ 4 \text{ if } L > 16 \end{cases}$$

where r is the price of one unit of capital. Assume for now that the optimal value of L is at most 16. If this is the case, the problem reads

$$\max_{L \geq 0} p\sqrt{L} - wL - 16r$$

whose first-order condition at an interior solution is:

$$\frac{p}{2\sqrt{L}} = w \Rightarrow L = \frac{p^2}{4w^2} \text{ and } \pi = \frac{p^2}{4w} > 0$$

This is indeed the solution if $L = \frac{p^2}{4w^2} \leq 16$, which implies that this is the solution if $\frac{w}{p} \geq \frac{1}{8}$. If $\frac{w}{p} < \frac{1}{8}$, the solution is $L = 16$.

In our case, if $w = 1$ and $p = 4$, the solution is $L = \frac{4^2}{4} = 4$.

(c) If $w = 1$ and $p = 10$, notice that $\frac{w}{p} < \frac{1}{8}$. For this reason, the firm chooses $L = 16$. If we ignored the change in the expression for the production function and tried to solve the problem $\max_{L \geq 0} p\sqrt{L} - wL - 16r$, the solution would be $L = 25$, which exceeds the threshold value of 16 where the expression for the short-run production function changes.

(d) From part b, if $\frac{w}{p} \geq \frac{1}{8}$ then $L = \frac{p^2}{4w^2}$. On the other hand, the firm demands an amount of labor equal 16 if $\frac{w}{p} < \frac{1}{8}$. Intuitively, if $\frac{w}{p} < \frac{1}{8}$, labor is very cheap, inducing the firm to choose as much labor as possible. The thing is that it does not make sense to use more than 16 units of labor, since there will be no further expansion in ouput. For this reason, the firm's short-run demand for labor is

$$L = \begin{cases} \frac{p^2}{4w^2} \text{ if } \frac{w}{p} \geq \frac{1}{8} \\ 16 \text{ if } \frac{w}{p} < \frac{1}{8} \end{cases}$$